

ON G -EQUIVARIANT MODULAR CATEGORIES

ALEXANDER KIRILLOV, JR.

1. INTRODUCTION

This paper is was born from an attempt to provide a suitable mathematical formalism for description of orbifold models of rational conformal field theory. Such models arise in the study of conformal field theories in which we have a finite group of automorphisms G of the vertex operator algebra V (or, in other terminology, the chiral algebra). In this case, we can form “quotient” theory which is described by the subalgebra of invariants $V^G \subset V$. These theories have been studied in numerous papers; see [DY] for references.

Vertex operator algebras are complicated objects, and working with them is not easy. However, it is well known that for usual (non-orbifold) theories, many features of the theory can be captured by a relatively simple algebraic formalism, namely that of tensor categories and modular functors (see, e.g., [T1], [BK] for an overview).

The goal of this paper is to define a notion of a G -equivariant modular tensor category that would generalize the above formalism to the theories with a finite group of automorphisms G and in particular, give a description of the orbifold theory in terms of the original theory. Our motivating example is the category of twisted modules over a VOA with an action of group of automorphisms G . We plan to continue this paper with a series of papers defining G -equivariant versions of modular functor, both in topological and complex-analytic formulations.

The starting point of this paper is the definition of G -equivariant fusion category due to Turaev [T2] (who used the name G -crossed category). This is a category with the action of a finite group G and with a G -grading. It can be shown that for a VOA V with an action of a finite group G , the category of twisted V -modules is a G -equivariant fusion category.

The main results of this paper are as follows:

- (1) For a given G -equivariant fusion category \mathcal{C} , we define the notion of “orbifold” category \mathcal{C}/G , which is a fusion category (this definition is not new), and study basic properties of this category. In the example when \mathcal{C} is the category of twisted modules over a VOA V , under some technical restrictions on V and V^G , the orbifold category is the category of V^G -modules. We show that this construction is equivalent to the approach based on the notion of algebra in a category (which, in the language of VOA’s, corresponds to considering the original VOA V as a module over V^G), developed in the series of papers [KO, K1, K2].
- (2) For a G -equivariant fusion category \mathcal{C} , we define the “extended” Verlinde algebra $\tilde{\mathcal{V}}(\mathcal{C})$ (which is no longer commutative) and give a simple description

- of the Verlinde algebra of the orbifold category $\mathcal{V}(\mathcal{C}/G)$ in terms of $\tilde{\mathcal{V}}(\mathcal{C})$ (Corollary 8.18).
- (3) We define the s, t -matrices for the extended Verlinde algebra $\tilde{\mathcal{V}}(\mathcal{C})$ and show that if s is non-degenerate, then these matrices define an action of the modular group $\mathrm{SL}_2(\mathbb{Z})$ on $\tilde{\mathcal{V}}(\mathcal{C})$; in this case, we call the category \mathcal{C} “modular G -equivariant category”. Note that this definition differs from that used by Turaev. We show that \mathcal{C} is modular iff the orbifold category \mathcal{C}/G is modular.
 - (4) We show that the s -matrix interchanges the two algebra structures on $\tilde{\mathcal{V}}$, the tensor product \otimes and “convolution product” $*$. This is an analogue of the statement that “ s -matrix diagonalizes the fusion rules”, which for usual fusion categories immediately gives the famous Verlinde formula for fusion coefficients. In the G -equivariant case, the situation is more complicated, as both \otimes and $*$ are non-commutative, so s -matrix does not exactly diagonalize the fusion rules; however, in some special cases (e.g., when G is commutative), one can indeed use this result to get some non-trivial results about the fusion coefficients of \mathcal{C} and \mathcal{C}/G . We plan to pursue this in subsequent papers.

The paper is organized as follows. In Section 2, we define our main object, G -equivariant fusion category, following [T2]. In Section 3, we explain the construction of “orbifold quotient” \mathcal{C}/G of a G -equivariant fusion category \mathcal{C} . This orbifold quotient is a (usual) fusion category; in the main example, when \mathcal{C} is the category of twisted modules over a VOA V , the orbifold quotient \mathcal{C}/G is the category of modules over V^G , which explains the name. In Section 4, we discuss the relation of this construction with the approach based on algebras in category developed in [KO, K1, K2].

Section 5 is devoted to examples of G -equivariant fusion categories and corresponding fusion categories; among other examples, it discusses category of (twisted) G -graded vector spaces, whose orbifold is the category of modules over (twisted) Drinfeld double $D^\omega(G)$, and the category of twisted modules over a VOA V .

Section 6 briefly reviews presentation of morphisms in a G -equivariant fusion category by “ G -colored” tangles, following [T2].

In sections 7 and 8, we define extended Verlinde algebra of a G -equivariant fusion category. This notion is a non-trivial generalization of the usual Verlinde algebra; it is motivated by modular functor approach: this generalized Verlinde algebra can be defined as a vector space associated to torus with no punctures. Unlike the usual case, this extended Verlinde algebra is non-commutative.

The main results of the paper are contained in sections 9 and 10, where we define s -matrix for a G -equivariant fusion category. Using this s -matrix, we define a modular G -equivariant category as a G -equivariant fusion category with invertible s . (Note that our definition is different from Turaev’s one.) We show that in any modular G -equivariant category one has a natural action of the modular group $\mathrm{SL}_2(\mathbb{Z})$ on the extended Verlinde algebra. We show that \mathcal{C} is a modular G -equivariant category iff the orbifold category \mathcal{C}/G is modular; in this case, the “untwisted sector” \mathcal{C}_1 is also modular.

Throughout this paper, G is a finite group.

2. EQUIVARIANT FUSION CATEGORIES

In this section, we give a definition of a G -equivariant tensor category. Recall that a *ribbon* category is a rigid balanced braided monoidal category \mathcal{C} . The words “rigid” and “balanced” mean that one has the contravariant duality functor $V \mapsto V^*$ satisfying certain properties, and that there exist functorial isomorphisms $\delta_V: V \rightarrow V^{**}$ compatible with the tensor product; details can be found in [BK]. For a braided category, defining δ_V is equivalent to defining a collection of functorial isomorphisms $\theta_V: V \rightarrow V$ (“twists”).

We will be mostly interested in the case when \mathcal{C} is a semisimple abelian category over \mathbb{C} , with finite-dimensional spaces of morphisms. In this case, we also assume that all functors appearing in the definition of a ribbon category are additive and \mathbb{C} -linear on morphisms, and that the unit object $\mathbf{1}$ is simple. Such ribbon categories are usually called *fusion* categories. However, many of our results are valid without the semisimplicity assumption.

For a fusion category \mathcal{C} , we denote by $\mathcal{V}(\mathcal{C})$ the complexified Grothendieck ring of \mathcal{C} (frequently also called the fusion algebra, or the *Verlinde algebra*). This is a commutative algebra over \mathbb{C} with the basis given by classes of simple objects; it will be discussed in detail in Section 7.

The following definition is due to Turaev [T2] (who used the term G -crossed category).

2.1. Definition. A G -equivariant category \mathcal{C} is an abelian category with the following additional structure:

G -grading: Decomposition

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$$

where each \mathcal{C}_g is a full subcategory in \mathcal{C} . We will call objects $V \in \mathcal{C}_g$ “ g -twisted”. In particular, objects $V \in \mathcal{C}_1$ will be called “neutral”; in physical literature, the subcategory \mathcal{C}_1 is usually called the “untwisted sector”

Action of G : For each $g \in G$, we are given a functor $R_g: \mathcal{C} \rightarrow \mathcal{C}$ and functorial isomorphisms $\alpha_{gh}: R_g \circ R_h \xrightarrow{\sim} R_{gh}$ such that $R_1 = \text{id}$, $R_g \mathcal{C}_h \subset \mathcal{C}_{ghg^{-1}}$, and $\alpha_{g_1 g_2, g_3} \circ \alpha_{g_1, g_2} = \alpha_{g_1, g_2 g_3} \circ \alpha_{g_2, g_3}$ (both sides are functorial isomorphisms $R_{g_1} R_{g_2} R_{g_3} \xrightarrow{\sim} R_{g_1 g_2 g_3}$).

Following [T2], we will also frequently use notation ${}^g V$ for $R_g(V)$.

A G -equivariant fusion category is a semisimple G -equivariant abelian category which in addition has the following structures:

- Structure of a rigid monoidal category such that
 - $\mathbf{1}$ is a simple object
 - R_g is a tensor functor
 - for $X \in \mathcal{C}_g, Y \in \mathcal{C}_h, X \otimes Y \in \mathcal{C}_{gh}$
- Functorial isomorphisms $\delta_V: V \rightarrow V^{**}$, satisfying the same compatibility conditions as in the absence of G (see [BK]) and additional condition $R_g(\delta_V) = \delta_{R_g(V)}$.
- A collection of functorial isomorphisms $R_{V,W}: V \otimes W \rightarrow {}^g W \otimes V$ for every $V \in \mathcal{C}_g, W \in \mathcal{C}_h$, satisfying an analog of the pentagon axiom (see [T2, Section 2.2]).

The definition immediately implies that $\mathbf{1} \in \mathcal{C}_1$ and that for $V \in \mathcal{C}_g$, $V^* \in \mathcal{C}_{g^{-1}}$. Also, since in a rigid category the unit object and dual is unique up to unique isomorphism, we have canonical identification

$$(2.1) \quad \begin{aligned} {}^g \mathbf{1} &= \mathbf{1}, \\ ({}^g V)^* &= {}^g(V^*) \end{aligned}$$

2.2. Remark. From now on, we will refer to associativity, unit, and δ morphism, as well as isomorphisms (2.1), and their compositions, as “canonical” isomorphisms, and we will omit them in the formulas, writing, e.g., $V_1 \otimes V_2 \otimes V_3$ rather than $(V_1 \otimes V_2) \otimes V_3$. Thus, all identities between morphisms written below only make sense after insertion of appropriate “canonical” morphisms. Pedantic reader may complete all computations by inserting appropriate canonical morphisms. See [BK, Section 1.1] for discussion of this.

Note that any abelian category \mathcal{C} admits a trivial G -grading, with $\mathcal{C}_1 = \mathcal{C}$, $\mathcal{C}_g = 0$ for $g \neq 1$. Thus, as a special case this definition includes fusion categories with action of G .

As usual, existence of morphism $\delta_V: V \rightarrow V^{**}$ is equivalent to a system of twists θ_V .

2.3. Lemma. *Let \mathcal{C} be a G -equivariant fusion category. Then one can define a collection of functorial morphisms $\theta_V: V \rightarrow {}^g V$, $V \in \mathcal{C}_g$, satisfying the following conditions:*

- (1) $\theta_{\mathbf{1}} = \text{id}$
- (2) $\theta_{U \otimes V} = (\theta \otimes \theta) R_{gV, U} R_{U, V}$
- (3) $\theta_{V^*} = R_{g^{-1}}(\theta_V^*)$
- (4) $\theta_{hV} = R_h(\theta_V)$

Conversely, δ_V can be recovered from θ , R , and monoidal structure.

The proof is completely parallel to the one in $G = \{1\}$ case (see, e.g., [BK, Section 2.2]).

From now on, \mathcal{C} will denote a G -equivariant fusion category.

The following lemma is an immediate consequence of the definition.

2.4. Lemma. *Let \mathcal{C} be a G -equivariant fusion category. Then the fusion algebra $\mathcal{V}(\mathcal{C})$ has a natural structure of a module over the Drinfeld double $D(G)$.*

A number of examples of G -equivariant fusion categories is given in Section 5. The most important of them is the example of twisted modules over a vertex operator algebra.

Note that a G -equivariant fusion category is *not* a fusion category: the braiding R does not satisfy the usual axioms of the commutativity isomorphism. However, for any G -equivariant fusion category \mathcal{C} , there are two related fusion categories. The first one is the “untwisted sector” $\mathcal{C}_1 \subset \mathcal{C}$ (in terminology of [T2], *neutral category*): it is easy to see from the definition that the equivariant fusion structure on \mathcal{C} , when restricted to \mathcal{C}_1 , defines a (usual) fusion category structure on \mathcal{C}_1 .

The second way to construct a fusion category from a G -equivariant fusion category is by “orbifolding”, or taking, in appropriate sense, the quotient by the action of G . This construction is studied in detail in the next section.

3. ORBIFOLD CATEGORY

Let \mathcal{C} be a G -equivariant fusion category. In this section we define the notion of “orbifold” category. This construction had appeared implicitly in [DY] and was explicitly defined in [K2], to which the reader is referred for proofs (where it is denoted by \mathcal{C}^G); in the case of “free” action of G , it is also described in [CM].

3.1. Definition. Let \mathcal{C} be a G -equivariant fusion category. Then the “orbifold fusion category” \mathcal{C}/G is a category with the following objects and morphisms:

- Objects: pairs $(X, \{\varphi\}_g)$, where $X \in \mathcal{C}$ and φ_g is a collection of \mathcal{C} -morphisms $\varphi_g: {}^gX \simeq X$ such that

$$(3.1) \quad \begin{aligned} \varphi_1 &= \text{id} \\ \varphi_g R_g(\varphi_h) &= \varphi_{gh} \end{aligned}$$

- Morphisms: $(X, \{\varphi_g\}) \rightarrow (Y, \{\psi_h\})$ are \mathcal{C} -morphisms $f: X \rightarrow Y$ such that $\psi_g \circ R_g(f) = f \circ \varphi_g$.

3.2. Remark. Condition (3.1) ensures that there is a canonical way of identifying all twists gX . Namely, if we denote, for $g, h \in G$,

$$(3.2) \quad \varphi_{g,h} = \varphi_g^{-1} \varphi_h: {}^hX \xrightarrow{\sim} {}^gX,$$

then (3.1) implies

$$(3.3) \quad \begin{aligned} \varphi_{g,h} \varphi_{h,f} &= \varphi_{g,f} \\ R_g \varphi_{a,b} &= \varphi_{ga,gb} \end{aligned}$$

Conversely, if $\varphi_{g,h}: {}^hX \xrightarrow{\sim} {}^gX$ is a system of isomorphisms satisfying (3.3), then $\varphi_g = \varphi_{1,g}$ satisfies (3.1).

3.3. Lemma. *The orbifold category \mathcal{C}/G is an abelian category.*

The proof of this lemma is straightforward (see [K2] for details).

It is immediate from the definition that \mathcal{C}/G has a natural structure of a module category over the category $\text{Rep } G$ of finite-dimensional complex G -modules: for a representation $\rho: G \rightarrow \text{End } V$, we define $\rho \otimes (X, \{\varphi\}) = (V \otimes X, \{\rho_g \otimes \varphi_g\})$ (see [O] for overview of the notion of module category).

The notation \mathcal{C}/G is justified by the following lemma:

3.4. Lemma. *If \mathcal{C} is semisimple, and G acts freely on the set $I(\mathcal{C})$ of isomorphism classes of simple objects, then \mathcal{C}/G is also semisimple, with the set of simple objects $I(\mathcal{C}/G) = I(\mathcal{C})/G$, and fusion algebra $\mathcal{V}(\mathcal{C}/G) = \mathcal{V}(\mathcal{C})^G$.*

However, in general it is not true that $\mathcal{V}(\mathcal{C}/G) = \mathcal{V}(\mathcal{C})^G$; instead, $\mathcal{V}(\mathcal{C}/G)$ can be described in terms of *extended* Verlinde algebra of \mathcal{C} (see Corollary 8.18).

The following theorem, proved in [DY] in the context of twisted modules over VOAs and in [K2] in the language of G -equivariant categories, gives a full description of \mathcal{C}/G as an abelian category.

For $i \in I(\mathcal{C})$, let $G_i = \text{Stab}(i) = \{g \in G \mid {}^gV_i \simeq V_i\}$. Note, however, that a priori we do not have a canonical isomorphism between gV_i and V_i . We can choose such an isomorphism $\lambda_g: {}^gV_i \simeq V_i$ arbitrarily; then we have $\lambda_g R_g(\lambda_h) = \alpha_{gh} \lambda_{gh}$ for some $\alpha: G_i^2 \rightarrow \mathbb{C}^\times$. It is easy to see that α is a two-cocycle, and that the class $[\alpha] \in H^2(G_i, \mathbb{C}^\times)$ does not depend on the choice of λ_g . Thus, action of G by automorphisms of \mathcal{C} not only gives an action of G on the set $I(\mathcal{C})$, but also, for

every $i \in I(\mathcal{C})$, a cohomology class $[\alpha_i] \in H^2(G_i, \mathbb{C}^\times)$. Such a cohomology class defines a central extension

$$1 \rightarrow \mathbb{C}^\times \rightarrow \widehat{G}_i^\alpha \rightarrow G_i \rightarrow 1$$

Define the twisted group algebra $\mathbb{C}^{\alpha_i}[G_i]$ by

$$(3.4) \quad \mathbb{C}^{\alpha_i}[G_i] = \mathbb{C}[\widehat{G}_i^\alpha]/([c] - c[1])$$

where, for $c \in \mathbb{C}^\times$, we denote by $[c]$ the class of the corresponding element in the central extension $\mathbb{C}^\times \subset \widehat{G}_i^\alpha$. Then one easily sees that choosing a lifting $G_i \hookrightarrow \widehat{G}_i^\alpha$ gives a basis in $\mathbb{C}^\alpha[G]$ consisting of classes $[g], g \in G$ with multiplication law $[g][h] = \alpha_{gh}[gh]$. Thus, representations of $\mathbb{C}^{\alpha_i}[G_i]$ (or, equivalently, the category of representations of the central extension \widehat{G}_i^α such that an element $c \in \mathbb{C}^\times \subset \widehat{G}_i^\alpha$ acts by multiplication by c) are exactly the projective representations of G_i with the cocycle α .

3.5. Theorem ([DY, K2]). *As an abelian category, \mathcal{C}/G is equivalent to*

$$\bigoplus_{i \in I/G} \text{Rep } \mathbb{C}^{\alpha_i^{-1}}[G_i]$$

where i runs over the set of representatives of G -orbits in I .

Proof and details can be found in [DY, Theorem 3.5], [K2, Theorem 3.5].

3.6. Corollary. *\mathcal{C}/G is a semisimple abelian category.*

Indeed, it is known (see, e.g., [Kar]) that $\mathbb{C}^{\alpha_i}[G_i]$ is a semisimple associative algebra, and thus the category of representations is semisimple.

3.7. Corollary. *As a vector space, the Verlinde algebra $\mathcal{V} = \mathcal{V}(\mathcal{C}/G)$ is given by*

$$\mathcal{V} = \bigoplus_{i \in I/G} Z(\mathbb{C}^{\alpha_i^{-1}}[G_i])$$

where

$$Z(\mathbb{C}^{\alpha_i^{-1}}[G_i]) = (\mathbb{C}^{\alpha_i^{-1}}[G_i])^{G_i} = \{f \in \mathbb{C}^{\alpha_i^{-1}}[G_i] \mid [g]f[g]^{-1} = f \ \forall g \in G_i\}$$

is the center of $\mathbb{C}^{\alpha_i^{-1}}[G_i]$

Proof. Since $\mathbb{C}^{\alpha_i^{-1}}[G_i]$ is semisimple, $Z(\mathbb{C}^{\alpha_i^{-1}}[G_i]) = \bigoplus_\lambda \text{End}(V_\lambda)$, where V_λ are irreducible $\mathbb{C}^{\alpha_i^{-1}}[G_i]$ -modules. Thus, $\text{id}_{V_\lambda} \mapsto [V_\lambda]$ is an isomorphism between $Z(\mathbb{C}^{\alpha_i^{-1}}[G_i])$ and the Verlinde algebra of $\text{Rep } \mathbb{C}^{\alpha_i^{-1}}[G_i]$. \square

3.8. Remark. In fact, the isomorphism of Corollary 3.7 is actually an isomorphism of associative algebras if \mathcal{V} is considered as an algebra with respect to the convolution product, which will be discussed in detail in Section 7.

It turns out that the structure of G -equivariant fusion category on \mathcal{C} gives rise to structure of fusion category on \mathcal{C}/G .

3.9. Theorem. *Let \mathcal{C} be a G -equivariant fusion category. Then the orbifold category \mathcal{C}/G is itself a fusion category, with the following tensor product, unit object, and*

duality:

$$\begin{aligned} (X, \{\varphi_g\}) \otimes (Y, \{\psi_g\}) &= (X \otimes Y, \{\varphi_g \otimes \psi_g\}) \\ \mathbf{1}_{\mathcal{C}/G} &= (\mathbf{1}, \{\text{id}\}) \\ (X, \{\varphi_g\})^* &= (X^*, \{(\varphi_g^*)^{-1}\}) \end{aligned}$$

Equivalently, in terms of morphisms $\varphi_{g,h}$ defined in Remark 3.2, the corresponding isomorphisms for X^* are defined by $\varphi_{h,g}^*: {}^h X^* \xrightarrow{\sim} {}^g X^*$.

The associativity, unit, and balancing isomorphisms δ are inherited from \mathcal{C} and the braiding is defined by

$$X \otimes Y \xrightarrow{R} {}^g Y \otimes X \xrightarrow{\psi_g \otimes 1} Y \otimes X.$$

In this category, the universal twist $\theta: X \rightarrow X$ is defined as follows: if we write $X = \bigoplus_h X_h$, $X_h \in \mathcal{C}_h$, then θ is the direct sum of the following composition:

$$X_h \xrightarrow{\theta} {}^h X_h \xrightarrow{\varphi_h} X_h.$$

The proof is straightforward: the associativity, unit, and rigidity automorphisms are inherited from \mathcal{C} ; verification of all identities is left to the reader.

One has some natural functors relating categories \mathcal{C} and \mathcal{C}/G :

$$\begin{aligned} (3.5) \quad \text{Ind}: \mathcal{C} &\rightarrow \mathcal{C}/G & \text{Res}: \mathcal{C}/G &\rightarrow \mathcal{C} \\ V &\mapsto (X, \{\varphi_g\}) & (X, \{\varphi_g\}) &\mapsto X \\ X &= \bigoplus_h {}^h V & & \\ \varphi_g: \bigoplus_h {}^{gh} V &\rightarrow \bigoplus_h {}^h V & & \end{aligned}$$

where $\varphi_g: \bigoplus_h {}^{gh} V \rightarrow \bigoplus_h {}^h V$ is the permutation of summands.

3.10. Example. Let $V = \mathbf{1} \in \mathcal{C}$; then $\text{Ind } V = \bigoplus_g {}^g \mathbf{1}$. Using canonical isomorphism (2.1) to identify ${}^g \mathbf{1} = \mathbf{1}$, we can identify

$$(3.6) \quad \text{Ind } V = \bigoplus_g {}^g \mathbf{1} = \mathcal{F}(G) \otimes \mathbf{1},$$

where $\mathcal{F}(G)$ is the vector space of functions on G . One easily sees that under this identification, φ_g becomes the left regular action of G on $\mathcal{F}(G)$.

The functor Ind should be thought of as an analog of “induction” functor in usual representation theory; the functor Res , as the “restriction” functor (see Example 5.4 below).

The following theorem is an analogue of [KO, Theorem 1.6].

3.11. Theorem. (1) *The functors Ind and Res are adjoint to each other: for any $V \in \mathcal{C}$, $X \in \mathcal{C}/G$.*

$$(3.7) \quad \text{Hom}_{\mathcal{C}}(\text{Res } X, V) = \text{Hom}_{\mathcal{C}/G}(X, \text{Ind } V)$$

(2) *The functor Res is a tensor functor.*

(3) *Res is compatible with duality and balancing: $(\text{Res } V)^* = \text{Res}(V^*)$, $\text{Res}(\delta) = \delta$. In particular, $\dim_{\mathcal{C}} \text{Res } X = \dim_{\mathcal{C}/G} X$*

The proof is straightforward and left to the reader.

4. ORBIFOLDS AND ALGEBRAS IN A CATEGORY

The theory of orbifold categories is closely related to the approach based on algebras in category, discussed in [K1, K2, FSch]. For reader's convenience, we briefly review here some of the definitions and results of [KO, K1, K2]. Recall that if \mathcal{A} is a fusion category, a commutative algebra in \mathcal{A} is an object $A \in \mathcal{A}$ with multiplication morphism $\mu: A \otimes A \rightarrow A$ satisfying obvious axioms. We will always assume that A satisfies two additional properties:

$$(4.1) \quad \begin{aligned} \theta_A &= \text{id} \\ A &\text{ is rigid} \end{aligned}$$

The last condition means that $\mathbf{1}$ has multiplicity 1 in A and the composition $A \otimes A \xrightarrow{\mu} A \rightarrow \mathbf{1}$ is a non-degenerate pairing (see [KO] for details).

For each such algebra A , we can define the category of (left) A -modules, with objects being objects of \mathcal{A} with a morphism $\mu_V: A \otimes V \rightarrow V$. We denote the category of A -modules by $A\text{-Mod}$. It can be shown that $A\text{-Mod}$ is a semisimple abelian category (see [KO, Theorem 3.3]). This category has a structure of a monoidal category which in general is not braided. One also has natural functors $F: \mathcal{A} \rightarrow A\text{-Mod}, G: A\text{-Mod} \rightarrow \mathcal{A}$ (see [KO, Theorem 1.6]) which are adjoint to each other; the functor F is a tensor functor.

Let us consider a special case, when we are given an action of a finite group G by automorphisms π_g of A . Assume additionally that this action satisfies the following conditions:

$$(4.2) \quad \begin{aligned} &\text{The action is faithful: if } g \neq 1, \text{ then } \pi_g \neq \text{id} \\ &A^G = \mathbf{1}. \end{aligned}$$

4.1. Example. Let $\mathcal{A} = \text{Rep } G$ be the category of G -modules, and $A = \mathcal{F}(G)$ the algebra of functions on G , with pointwise multiplication and structure of G module given by left regular action of G on $\mathcal{F}(G)$. Let π_g be the *right* regular action of G on $\mathcal{F}(G)$; it commutes with the left regular action and thus defines an algebra automorphism. One easily sees that this action satisfies properties (4.1, 4.2).

For an algebra with action of G by automorphisms, we can define the notion of g -twisted module: an A -module is called g -twisted if $\mu_V \circ R^2 = \mu_V \circ (\pi_{g^{-1}} \otimes \text{id})$.

The following theorem summarizes many of the results on [K1, K2].

4.2. Theorem. *Let \mathcal{A} be a fusion category and A —a commutative algebra in \mathcal{A} satisfying conditions (4.1), with an action of a finite group G satisfying conditions (4.2). Then:*

- (1) [K1, Theorems 2.11, 2.15] *\mathcal{A} contains as a full subcategory the category $\text{Rep } G$. The algebra A lies in this subcategory and can be identified with the algebra $\mathcal{F}(G)$ of Example 4.1.*
- (2) [K2, Section 5] *The category $\mathcal{C} = A\text{-Mod}$ has a natural structure of a G -equivariant fusion category, with \mathcal{C}_g being the subcategory of g -twisted A -modules.*
- (3) [K2, Theorems 4.1, 4.4] *The category \mathcal{A} is naturally equivalent to the orbifold $(A\text{-Mod})/G$. Under this equivalence, the functors $F: \mathcal{A} \rightarrow A\text{-Mod}, G: A\text{-Mod} \rightarrow \mathcal{A}$ are identified with the functors Res, Ind defined by (3.5).*

It turns out that this result can be reversed.

4.3. Theorem. *Let \mathcal{C} be a G -equivariant fusion category, $\mathcal{A} = \mathcal{C}/G$; by Theorem 3.9, \mathcal{A} is a fusion category. Let $A = \text{Ind}(\mathbf{1}_{\mathcal{C}}) \in \mathcal{A}$. Then A has a natural structure of a commutative algebra in \mathcal{A} with an action of G , satisfying conditions (4.1), (4.2), and the category of A -modules is naturally equivalent to the category \mathcal{C} .*

Proof. By Example 3.10, as an object of \mathcal{C}/G , $A = \mathcal{F}(G) \otimes \mathbf{1}_{\mathcal{C}/G}$, where $\mathcal{F}(G)$ is the algebra of functions on G . Define multiplication on A and action of G by automorphisms on A as in Example 4.1. Then it is immediate that A is a commutative algebra in \mathcal{A} , satisfying conditions (4.1), (4.2).

One can also explicitly describe the category of A -modules. Namely, writing $A = \bigoplus^g \mathbf{1}$, we see that an A -module in the category $\mathcal{A} = \mathcal{C}/G$ is an object $(X, \{\varphi_g\}) \in \mathcal{C}/G$ along with a collection of \mathcal{C} -morphisms $\mu_g: X \rightarrow X$ such that $\mu_{g_1} \mu_{g_2} = \delta_{g_1, g_2} \mu_{g_1}$, $\sum \mu_g = \text{id}$ and

$$\varphi_g R_g \mu_h = \mu_{gh} \varphi_g.$$

To show that the category of A -modules is equivalent to \mathcal{C} , construct the functors $\mathcal{C} \rightarrow A\text{-Mod}$, $A\text{-Mod} \rightarrow \mathcal{C}$ as follows:

$\mathcal{C} \rightarrow A\text{-Mod}$: for $V \in \mathcal{C}$, consider $\text{Ind } V = (\bigoplus^g V, \{\varphi_g\}) \in \mathcal{C}/G$ and define on it the structure of A -module by defining $\mu_g = \text{id}_V$. It is easy to check that it satisfies all the required properties.

$A\text{-Mod} \rightarrow \mathcal{C}$: let $(X, \{\varphi_g\}) \in \mathcal{C}/G$, with structure of A -module given by $\mu_g: X \rightarrow X$. Define an object $V \in \mathcal{C}$ by $V = \text{Im } \mu_1$.

It is trivial to check that these two functors are inverse to each other, and thus establish an equivalence of categories $\mathcal{C} \simeq A\text{-Mod}$. \square

Thus, starting from a fusion category \mathcal{A} with a commutative algebra A satisfying (4.1), (4.2), one can construct a G -equivariant fusion category $\mathcal{C} = A\text{-Mod}$. The original category \mathcal{A} can be recovered as $\mathcal{A} = \mathcal{C}/G$. Conversely, starting from a G -equivariant fusion category \mathcal{C} , one can define the fusion category $\mathcal{A} = \mathcal{C}/G$ and an algebra $A \in \mathcal{A}$ with an action of G ; then \mathcal{C} can be recovered as the category of A -modules. This allows us to use some results about the category of A -modules to answer questions about relation between \mathcal{C} and \mathcal{C}/G .

5. EXAMPLES OF EQUIVARIANT FUSION CATEGORIES

5.1. Example. Let $\mathcal{C} = \mathcal{V}ec$ be the category of vector spaces, with trivial grading and trivial action of G : $R_g = \text{id}$ for all g . Then the corresponding orbifold category $\mathcal{V}ec/G$ is the category $\text{Rep } G$ of finite-dimensional G -modules, with the usual tensor product. The functors Ind and Res are given by $\text{Ind}(V) = V \otimes \mathcal{F}(G)$, where $\mathcal{F}(G)$ is the space of functions on G with left regular action of G , and $\text{Res}(X) = X$ is the forgetful functor.

5.2. Example. Let $\mathcal{C} = G\mathcal{V}ec$ be the category of G -graded vector spaces. It can be explicitly described as the category with simple objects $X_g, g \in G$, and tensor product, duality given by $X_g \otimes X_h = X_{gh}, X_g^* = X_{g^{-1}}$. Define the action of G by $R_g X_h = X_{ghg^{-1}}$. Then orbifold category $G\mathcal{V}ec/G$ is the category of finite-dimensional modules over the Drinfeld double $D(G)$ with the usual tensor product. In this case, $\text{Res}: \text{Rep } D(G) \rightarrow G\mathcal{V}ec$ is the usual forgetful functor, and $\text{Ind}: G\mathcal{V}ec \rightarrow \text{Rep } D(G)$ is the usual induction functor $\mathcal{V}ec \rightarrow \text{Rep } G$ with suitably

defined G -grading. This example is discussed in slightly different language in [K1, Sections 5,6].

5.3. Example. Let \mathcal{C} be a twisted category of G -graded vector spaces, i.e. a rigid monoidal category which coincides with $G\mathcal{V}ec$ as an abelian category, and has tensor product, duality defined so that $X_g \otimes X_h \simeq X_{gh}$, $X_g^* \simeq X_{g^{-1}}$ (non-canonically). Each such category defines a 3-cocycle $\omega \in C^3(G, \mathbb{C}^\times)$: if we choose a system of isomorphisms $\alpha_{gh}: X_g \otimes X_h \xrightarrow{\sim} X_{gh}$, then $\alpha_{g_1, g_2 g_3} \alpha_{g_2, g_3} = \omega(g_1, g_2, g_3) \alpha_{g_1 g_2, g_3} \alpha_{g_1, g_2}$. It is easy to show that two such categories are equivalent as monoidal categories iff $[\omega] = [\omega']$, and that this defines a bijection between equivalence classes of twisted categories of G -graded vector spaces and $H^3(G, \mathbb{C}^\times)$. We will denote by $G\mathcal{V}ec^\omega$ the twisted category of G -graded vector spaces with cocycle ω . (A slightly different but equivalent description of these categories is given in [T2, Section 1.3, Section 2.6]).

Define the action of G by $R_g X = X_g \otimes X \otimes X_g^*$, and the braiding isomorphism as composition

$$X_g \otimes X_h \xrightarrow{\sim} X_g \otimes X_h \otimes X_g^* \otimes X_g = {}^g X_h \otimes X_g.$$

One easily sees that this defines on $G\mathcal{V}ec^\omega$ a structure of G -equivariant fusion category. In this case, the corresponding orbifold category $G\mathcal{V}ec^\omega/G$ can be shown to coincide with the category of modules over twisted Drinfeld double $D^\omega(G)$ as defined in [DPR], [DN].

Note that unlike the construction in [T2, Section 2.6], our definition does not require that the cocycle ω be G -invariant. On the other hand, our construction is not the most general: e.g., there are different ways to define braiding in $G\mathcal{V}ec^\omega$; see [T2] for details.

5.4. Example. Let N be a group on which G acts by automorphisms. Let $\mathcal{C} = \text{Rep } N$ be the category of finite-dimensional N -modules. Define on this category an action of G as follows: for a module M , we let $R_g(M)$ be the same vector space as M but with the action of M defined by $\rho_{R_g(M)}(n) = \rho_M(g^{-1}(n))$. Then \mathcal{C} becomes an equivariant fusion category (with trivial G -grading), and the corresponding orbifold category is $\text{Rep } N/G = (G \ltimes N) - \text{Mod}$ (see [K2, Theorem 2.1]). In this case, the functors $\text{Res}: \text{Rep}(G \ltimes N) \rightarrow \text{Rep } N$ and $\text{Ind}: \text{Rep } N \rightarrow \text{Rep}(G \ltimes N)$ are the usual induction and restriction functors.

5.5. Example. Let V be a rational vertex operator algebra such that the category of V -modules is a fusion category, and let G be a finite group subgroup of automorphisms of V . Then we can define the category $\mathcal{C} = V - \text{Mod}_{tw}$ of twisted V -modules as in [DVVV], or, in more detail, in [DLM]. This category by definition has a G -grading: $V - \text{Mod}_{tw} = \bigoplus_g V - \text{Mod}_g$, with $V - \text{Mod}_1 = V - \text{Mod}$. For a twisted V -module M , let $R_g(M)$ coincide with M as a vector space but define the action of V by $Y_{R_g(M)}(v, z) = Y_M(g^{-1}(v), z)$. Then the category $V - \text{Mod}_{tw}$ has a natural structure of a G -equivariant category. Under suitable assumptions on V , this category has a natural structure of G -equivariant fusion category. Indeed, as shown in [KO, K1], the category of twisted modules over V is equivalent to the category of A -modules, where A is V considered as an associative commutative algebra in the category $\mathcal{A} = V^G - \text{Mod}$ of moduleks over V^G . Thus, applying the results of Section 4, we see that $V - \text{Mod}_{tw}$ is a G -equivariant fusion category. As a corollary of Theorem 4.2, we see that the orbifold category in this case is $\mathcal{A} = \mathcal{C}/G$, i.e., $V^G - \text{Mod} = V - \text{Mod}/G$.

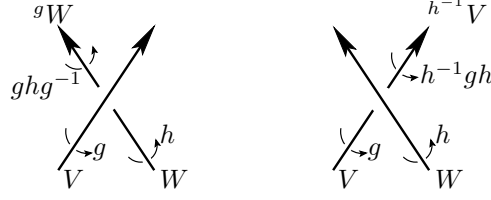


FIGURE 1. Crossing

This example is the main motivation for the study of G -equivariant fusion categories.

5.6. *Remark.* It would be interesting to give a direct description of the monoidal structure on the category of twisted modules over V , i.e. a description which does not use restriction of the modules to V^G . The only result in this direction we were able to find is the paper [G], so many details are still missing.

6. GRAPHICAL DESCRIPTION OF MORPHISMS IN G -EQUIVARIANT CATEGORIES

In this section, we briefly review the graphical technique for representing morphisms in a G -equivariant fusion category, generalizing well-known graphical technique for representing morphisms in a braided tensor category by tangles. The results of this section are due to Turaev [T2]. For now, we present a very simplified description; a more detailed description, using the language of links and tangles in a 3-manifold with a principal G -bundle, can be found in [T2].

Recall that a *tangle diagram* is a collection of oriented arcs and circles in $\mathbb{R} \times [0, 1]$, where the arcs ends must lie on the lines $\mathbb{R} \times \{0\}, \mathbb{R} \times \{1\}$. The only intersections allowed are transversal double intersections, and for each such intersection, one of the strands is specified as “top”, and the other as “bottom”. Such tangle diagrams naturally arise as projections of tangles in \mathbb{R}^3 , and it is well-known that two diagrams correspond to isotopic tangles iff they can be obtained one from another by a sequence of Reidemeister moves.

This can be generalized to G -equivariant situation. For a tangle diagram T , a *segment* of T is a part of an arc or a circle between two undercrossings (i.e., points where a given arc goes under one of the other arcs).

6.1. **Definition.** Let \mathcal{C} be a G -equivariant category, and T — a tangle diagram. A \mathcal{C} coloring of T is an assignment to every segment of an arc or circle a pair $(g, V), g \in G, V \in \mathcal{C}_g$ (a *color*) satisfying the two conditions below. In the figures, we will show a color by writing the object V next to the segment, and writing g next to an arrow going under the corresponding strand, as in Figure 1. Since V determines g , we will frequently write just V and omit notation of g .

- (1) For every circle, the ordered product $\prod_i g_i = 1$, where the product is over all undercrossings of this circle, and g_i are the colors of crossing strand.
- (2) For two segments separated by an undercrossing, the colors are related as shown in Figure 1.

6.2. *Remark.* Condition (1) ensures compatibility of condition (2). Indeed, condition (2) implies that with each undercrossing, V is replaced by ${}^g V$. Thus, if we have a circle, condition (2) implies that ${}^g V = V$, where $g = \prod_i g_i$. However, in a tensor category, condition ${}^g V = V$ is almost never satisfied; at best, one could

expect that gV is isomorphic to V , but then we would need to specify explicitly the choice of isomorphism, since there is no canonical isomorphism. To avoid these problems, we require that $\prod g_i = 1$. More general situation, with $g \neq 1$ and a choice of isomorphism ${}^gV \xrightarrow{\sim} V$, is best dealt with by allowing circles with coupons described below.

It can be shown (see [T2, Lemma 3.2.1]) that a coloring is uniquely determined by specifying the color of just one segment on every arc and circle of T (these colors can not be chosen arbitrarily: condition (2) imposes restrictions on them). Therefore, in many figures we will specify the color at just one point on each circle or arc.

We have an obvious action of G on the set of coloring of a given tangle diagram T : for a color (V, h) , we define action of g by $R_g(V, h) = ({}^gV, ghg^{-1})$. One easily checks that this preserves conditions (1), (2).

As in the non-equivariant case, we can assign morphisms to tangle diagrams as follows. Let T be a tangle diagram. Then bottom of T defines a sequence of triples (g, V, ε) , where (g, V) is the color of the corresponding segment of T , and $\varepsilon = \pm$ is defined by the direction of the corresponding segment: $\varepsilon = +$ if it is directed up, $\varepsilon = -$ if it is directed down. Define

$$(6.1) \quad X_{in}(T) = \bigotimes V_i^{\varepsilon_i},$$

where the tensor product is over all ends of the arcs at the bottom of T , in the natural order (left to right), and $V^\varepsilon = V$ for $\varepsilon = +$, $V^\varepsilon = V^*$ for $\varepsilon = -$. If the bottom of T is empty, we let $X_{in}(T) = \mathbf{1}$. In a similar way, we define $X_{out}(T)$ by taking the product over the top of the diagram T .

6.3. Theorem. *Let \mathcal{C} be a G -equivariant fusion category, Then there is a unique way to assign to every colored tangle diagram T a \mathcal{C} -morphism $F(T): X_{in}(T) \rightarrow X_{out}(T)$ so that the following properties are satisfied:*

- (1) *For elementary crossing, “cap” and “cup” diagrams, $F(T)$ is the commutativity morphism R , rigidity morphism $V \otimes V^* \rightarrow \mathbf{1}$, and $\mathbf{1} \rightarrow V \otimes V^*$ respectively.*
- (2) *$F(T_1 \otimes T_2) = F(T_1) \otimes F(T_2)$, and $F(T_1 \circ T_2) = F(T_1)F(T_2)$, where $T_1 \otimes T_2$ is the diagram obtained by placing T_2 to the right of T_1 , and $T_1 \circ T_2$ is the diagram obtained by placing T_1 on top of T_2 (it is defined only if $X_{out}(T_2) = X_{in}(T_1)$).*

So defined assignment F satisfies the following properties:

G -equivariance: *For any $g \in G$,*

$$F(R_g T) = R_g(F(T)): {}^g X_{in}(T) \rightarrow {}^g X_{out}(T)$$

Independence of the choice of projection: *$F(T)$ is invariant under the Reidemeister moves for framed tangles shown in Figure 2.*

Orientation reversal: *$F(T)$ is invariant under simultaneously reversing direction of a component of T and replacing its color (V, g) by (V^*, g^{-1}) .*

Tensor product: *Replacing in T a component with color $(V_1 \otimes V_2, g_1 g_2)$ by two components, obtained by doubling the original component, and with colors (V_1, g_1) and (V_2, g_2) , does not change $F(T)$.*

The condition of invariance under Reidemeister moves means that $F(T)$ depends only on the isotopy class of the tangle and not on the choice of diagram representing

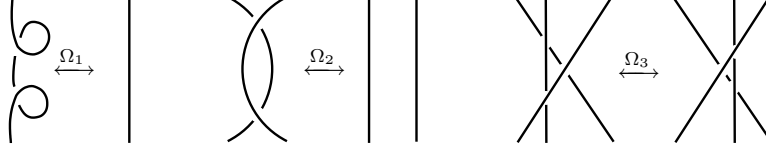


FIGURE 2. Reidemeister moves

this tangle. However, accurate explanation of this requires that we introduce the notion of \mathcal{C} -colored tangle which takes some time; see [T2].

6.4. Example. For tangle T shown in Figure 3 with $V \in \mathcal{C}_g$, $F(T) = \theta_V : V \rightarrow {}^gV$ (which explains why θ is called “twist”). For technical reasons, however, we will usually just draw box labeled by θ instead of drawing the twist.



FIGURE 3. Graphical representation of the universal twist

These results can naturally be extended to graphs with coupons, i.e. rectangular boxes; the color of such a coupon should be a morphism $\Phi : X_{in} \rightarrow X_{out}$, where X_{in}, X_{out} are tensor products of colors at the bottom (respectively, top) of the coupon. So defined invariants for graphs with coupons satisfy, in addition to the Reidemeister moves above, the moves shown in Figure 4. See [T2] for proofs and details.

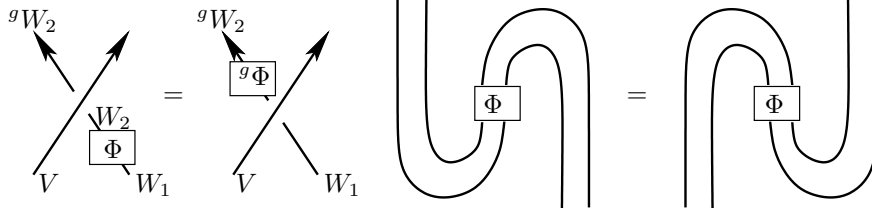


FIGURE 4.

7. VERLINDE ALGEBRA 1

In this section we recall known facts about the Verlinde algebra of a fusion category \mathcal{C} . All of them are well-known; however, they are formulated here in somewhat unusual form for the convenience of later generalization to the G -equivariant case, which will be done in the next section.

Throughout this section, \mathcal{C} is a fusion category. For simplicity, we assume that the set I of isomorphism classes of simple objects in \mathcal{C} is finite (all the results are actually valid without this assumption if we allow some of the objects we consider to be infinite sums of simple objects in \mathcal{C}).

Definition of \mathcal{V} . We define complex vector space $\mathcal{V} = \mathcal{V}(\mathcal{C})$ by

$$(7.1) \quad \mathcal{V} = \bigoplus_{i \in I} \text{Hom}(V_i, V_i),$$

where the sum is over the set $I = I(\mathcal{C})$ of equivalence classes of simple objects in \mathcal{C} . Then \mathcal{V} has a natural basis $\chi_i = [\text{id}_{V_i}]$.

This vector space also has a more invariant definition.

7.1. Theorem. \mathcal{V} is isomorphic to the vector space spanned by classes $[\varphi]$, where $\varphi: V \rightarrow V, V \in \mathcal{C}$, with the following relations

(1) For any $\lambda \in \mathbb{C}, \varphi, \psi \in \text{End } V$, one has

$$(7.2) \quad \lambda[\varphi] = [\lambda\varphi], \quad [\varphi + \psi] = [\varphi] + [\psi].$$

(2) For any $\varphi \in \text{End } V$ and isomorphism $f: V \xrightarrow{\sim} V'$, one has

$$(7.3) \quad [f\varphi f^{-1}] = [\varphi]$$

(3) If $W = \bigoplus W_i$, for some $W_i \in \mathcal{C}$, and $\varphi: W \rightarrow W, \varphi = \sum \varphi_{ij}, \varphi_{ij}: W_i \rightarrow W_j$, then

$$(7.4) \quad [\varphi] = \sum [\varphi_{ii}]$$

Proof. Denote temporarily by \mathcal{V}' the vector space generated by $[\varphi]$ with relations (7.2)–(7.4). Define a map $\mathcal{V}' \rightarrow \mathcal{V}$ as follows. Let $\varphi: V \rightarrow V$; write $V = \bigoplus H_i \otimes V_i$, where $H_i = \text{Hom}(V_i, V)$ are multiplicity spaces. Then $\varphi = \bigoplus \varphi_i \otimes \text{id}_{V_i}$, where $\varphi_i: H_i \rightarrow H_i$ is a linear map. We define the map $\mathcal{V}' \rightarrow \mathcal{V}$ by

$$(7.5) \quad [\varphi] \mapsto \sum (\text{tr } \varphi_i) \chi_i.$$

It is easy to see that relations (7.2)–(7.4) are satisfied, so this gives a well-defined map $\mathcal{V}' \rightarrow \mathcal{V}$. This map is clearly surjective. It is also surjective: if $\text{tr } \varphi_i = 0$ for all i , then it follows from (7.4) that $[\varphi] = 0$ in \mathcal{V}' . Thus, this map gives an isomorphism $\mathcal{V}' \xrightarrow{\sim} \mathcal{V}$. \square

From now on, we will frequently use this theorem, writing various operations in \mathcal{V} in terms of classes $[\varphi]$. Of course, whenever we define something in terms of classes $[\varphi]$, we need to verify that relations (7.2)–(7.4). This is usually trivial and therefore we will not write it explicitly.

In particular, for an object V we define $[V] = [\text{id}_V] = \sum (\dim H_i) \chi_i$. This brings us back to the standard definition of \mathcal{V} as the complexified Grothendieck ring of \mathcal{C} .

Tensor product in \mathcal{V} . The vector space \mathcal{V} has a structure of associative commutative algebra, which we will denote by \otimes . It is defined by

$$(7.6) \quad [\varphi] \otimes [\psi] = [\varphi \otimes \psi]$$

In particular,

$$\chi_i \otimes \chi_j = \sum N_{ij}^k \chi_k$$

where N_{ij}^k are fusion coefficients: $V_i \otimes V_j \simeq \sum N_{ij}^k V_k$.

The unit with respect to \otimes is $\chi_0 = [\mathbf{1}]$.

Convolution product in \mathcal{V} . There is also another associative, commutative product in \mathcal{V} , which we will denote by $*$ (it is sometimes called “the convolution product”). It is defined as follows: for $\varphi: V_i \rightarrow V_i, \psi: V_j \rightarrow V_j$, we define

$$(7.7) \quad [\varphi] * [\psi] = \delta_{ij} d_i^{-1} [\varphi\psi]$$

where

$$(7.8) \quad d_i = \dim V_i.$$

Recall that the numbers d_i are always non-zero (see, e.g. [BK, Section 2.4]).

In particular,

$$(7.9) \quad \chi_i * \chi_j = d_i^{-1} \delta_{ij} \chi_i.$$

The unit with respect to $*$ is

$$(7.10) \quad \mathbf{d} = \sum_{i \in I} d_i \chi_i.$$

Bilinear form. The Verlinde algebra \mathcal{V} also has a natural bilinear form. Namely, for $\varphi: V \rightarrow V$, let $\varphi^*: V^* \rightarrow V^*$ be the adjoint morphism. This defines an algebra automorphism $V \rightarrow V$. Also, define the “constant term” of an element $x \in \mathcal{V}$ by $[\chi_i]_0 = 0, i \neq 0$ and $[\chi_0]_0 = 1$. Then we define the bilinear form on \mathcal{V} by

$$(7.11) \quad (\varphi, \psi) = [\varphi \otimes \psi^*]_0.$$

7.2. Lemma. *The bilinear form (7.11) has the following properties:*

- (1) *It is symmetric and non-degenerate*
- (2) $(\chi_i, \chi_j) = \delta_{ij}$
- (3) $(x \otimes y, z) = (x, z \otimes y^*)$.

The proof of this lemma is straightforward and left to the reader.

Dimension homomorphism. We define the dimension homomorphism $d: \mathcal{V} \rightarrow \mathbb{C}$ by letting, for $\varphi: V \rightarrow V$,

$$(7.12) \quad d([\varphi]) = \text{tr}_V(\varphi).$$

Thus, $d[V] = \dim V$, so $d(\chi_i) = \dim(V_i) = d_i$. One easily sees that the dimension map has the following properties:

$$(7.13) \quad \begin{aligned} d(x \otimes y) &= d(x)d(y), \\ d(x^*) &= d(x), \\ d(\mathbf{1}) &= 1. \end{aligned}$$

7.3. Lemma. *Let $\mathbf{d} \in \mathcal{V}$ be defined by (7.10). Then*

- (1) $(\mathbf{d}, x) = d(x)$
- (2) $\mathbf{d}^* = \mathbf{d}$.
- (3) *For any $x \in \mathcal{V}$, $\mathbf{d} \otimes x = d(x)\mathbf{d}$.*

Proof. (1) is immediate from Lemma 7.2 and definition of \mathbf{d} ; (2) follows from (1) and (7.13). To prove (3), note that it suffices that both sides have the same inner product with any $y \in \mathcal{V}$. Using results of Lemma 7.2 and (7.13), we can write

$$(\mathbf{d} \otimes x, y) = (\mathbf{d}, y \otimes x^*) = d(y \otimes x^*) = d(y)d(x) = (d(x)\mathbf{d}, y).$$

□

7.4. Lemma. *For $x, y \in \mathcal{V}$, one has*

$$(x, y) = d(x * y).$$

Proof. Suffices to prove this for $x = \chi_i, y = \chi_j$, in which case $d(\chi_i * \chi_j) = \delta_{ij} d_i^{-1} d(\chi_i) = \delta_{ij}$. \square

7.5. Corollary. *If V is a simple module, and $\varphi, \psi \in \text{End}(V)$, then*

$$(7.14) \quad ([\varphi], [\psi]) = \frac{1}{\dim V} \text{tr}(\varphi\psi).$$

Note that this may be false if V is not simple: for example, if \mathcal{C} is the category of vector spaces, then $([\varphi], [\psi]) = (\text{tr } \varphi)(\text{tr } \psi)$, which in general is not equal to $\frac{1}{\dim V} \text{tr}(\varphi\psi)$.

Notice that there is a nice symmetry between \otimes and $*$:

$$\begin{array}{lll} \mathbf{1} \otimes x = x & \mathbf{d} \otimes x = d(x)\mathbf{d} & (x, y) = [x \otimes y^*]_0 \\ \mathbf{1} * x = [x]_0 \mathbf{1} & \mathbf{d} * x = x & (x, y) = d(x * y) \end{array}$$

This symmetry interchanges $\mathbf{1}$ with \mathbf{d} and $[\]_0$ with d .

8. VERLINDE ALGEBRA 2

In this section we give a definition of the extended Verlinde algebra for G -equivariant fusion categories, following the same steps as in Section 7 with suitable changes. The results of this section are new.

Definition of $\tilde{\mathcal{V}}$.

8.1. Definition. Let \mathcal{C} be a G -equivariant fusion category, and V_i —representatives of isomorphism classes of simple objects in \mathcal{C} . Then the extended Verlinde algebra of \mathcal{C} is defined by

$$\tilde{\mathcal{V}}(\mathcal{C}) = \bigoplus_{i \in I, g \in G} \text{Hom}_{\mathcal{C}}(V_i, {}^g V_i).$$

This definition is motivated by modular functor point of view: $\tilde{\mathcal{V}}(\mathcal{C})$ is the vector space assigned to a torus with no boundary components (see [T2, Section 8.6]).

From now on, we assume that the set I of isomorphism classes of simple objects in \mathcal{C} is finite, so that $\tilde{\mathcal{V}}$ is finite-dimensional.

Note that if $V_i \in \mathcal{C}_h$, then $\text{Hom}(V_i, {}^g V_i) = 0$ unless $ghg^{-1} = h$. Thus, the Verlinde algebra can be written as follows:

$$(8.1) \quad \begin{aligned} \tilde{\mathcal{V}}(\mathcal{C}) &= \bigoplus_{g, h: gh=hg} \tilde{\mathcal{V}}_{g,h}(\mathcal{C}) \\ \tilde{\mathcal{V}}_{g,h}(\mathcal{C}) &= \bigoplus_{i \in I_h} \text{Hom}(V_i, {}^g V_i) \end{aligned}$$

where I_h is the set of isomorphism classes of simple objects in \mathcal{C}_h . In particular,

$$\tilde{\mathcal{V}}_{1,*} = \bigoplus_{i \in I} \text{Hom}(V_i, V_i) = \mathcal{V}(\mathcal{C})$$

is the usual Verlinde algebra, i.e. the complexified Grothendieck ring of the category \mathcal{C} .

8.2. *Remark.* If the action of the group G on the set of isomorphism classes of simple objects is free, then it immediately follows from the definition that $\tilde{\mathcal{V}}_{g,*} = 0$ for $g \neq 1$, and thus $\tilde{\mathcal{V}} = \mathcal{V}$, i.e. the extended Verlinde algebra coincides with the usual Verlinde algebra.

As before, $\tilde{\mathcal{V}}$ has a more invariant definition.

8.3. **Theorem.** *Let $g, h \in G, gh = hg$. Then $\tilde{\mathcal{V}}_{g,h}$ is isomorphic to the vector space spanned by classes $[\varphi]$, where $\varphi: V \rightarrow {}^gV, V \in \mathcal{C}_h$, with the following relations*

(1) *For any $\lambda \in \mathbb{C}, \varphi, \psi: V \rightarrow {}^gV$, one has*

$$(8.2) \quad \lambda[\varphi] = [\lambda\varphi], \quad [\varphi + \psi] = [\varphi] + [\psi].$$

(2) *For any $\varphi: V \rightarrow {}^gV$ and isomorphism $f: V \xrightarrow{\sim} V'$, one has*

$$(8.3) \quad [R_g(f) \varphi f^{-1}] = [\varphi].$$

(3) *If $W = \bigoplus W_i$, for some $W_i \in \mathcal{C}_h$, and $\varphi: W \rightarrow W, \varphi = \sum \varphi_{ij}, \varphi_{ij}: W_i \rightarrow {}^gW_j$, then*

$$(8.4) \quad [\varphi] = \sum [\varphi_{ii}].$$

Proof. The proof is parallel to the proof of Theorem 7.1, with the following modification: the map $\tilde{\mathcal{V}}' \rightarrow \tilde{\mathcal{V}}$ is defined by

$$[\varphi] \mapsto \sum (\text{tr } \varphi_{ii}) [\lambda_{ii}]$$

where $\varphi: V \rightarrow {}^gV, V = \bigoplus H_i \otimes V_i, H_i = \text{Hom}(V_i, V)$, and we write $\varphi = \bigoplus \varphi_{ij} \otimes \lambda_{ij}, \varphi_{ij}: H_i \rightarrow H_j, \lambda_{ij}: V_i \rightarrow {}^gV_j$. \square

From now on, we will frequently use this theorem, writing various operations in $\tilde{\mathcal{V}}$ in terms of classes $[\varphi]$. Of course, whenever we define something in terms of classes $[\varphi]$, we need to verify that relations (8.2)–(8.4) are satisfied. This is usually trivial and therefore we will not write it explicitly.

In particular, for any $V \in \mathcal{C}$ we define $[V] = [\text{id}_V] \in \tilde{\mathcal{V}}$ and denote $\chi_i = [V_i] \in \tilde{\mathcal{V}}_{1,*}$. The elements χ_i form a basis in $\tilde{\mathcal{V}}_{1,*}$.

We have an obvious action of G on $\tilde{\mathcal{V}}$, given by

$$(8.5) \quad \begin{aligned} R_x: \tilde{\mathcal{V}}_{g,h} &\rightarrow \tilde{\mathcal{V}}_{xgx^{-1}, xhx^{-1}} \\ [\varphi] &\mapsto [R_x \varphi]. \end{aligned}$$

Tensor product in $\tilde{\mathcal{V}}$. The vector space $\tilde{\mathcal{V}}$ has a structure of associative algebra, which we will denote by \otimes . It is defined as follows: if $[\varphi] \in \tilde{\mathcal{V}}_{g_1, h_1}, [\psi] \in \tilde{\mathcal{V}}_{g_2, h_2}$, then

$$(8.6) \quad [\varphi] \otimes [\psi] = \begin{cases} [\varphi \otimes \psi] \in \tilde{\mathcal{V}}_{g_1, h_1 h_2} & g_1 = g_2 \\ 0 & g_1 \neq g_2 \end{cases}$$

8.4. **Lemma.** (1) *The product \otimes defined by (8.6) defines on $\tilde{\mathcal{V}}$ a structure of associative algebra with unit*

$$(8.7) \quad \tilde{\mathbf{1}} = \sum \chi_0^g,$$

where $\chi_0^g: \mathbf{1} \rightarrow {}^g\mathbf{1}$ is the canonical isomorphism (2.1).

(2) For $\varphi \in \tilde{\mathcal{V}}_{g_1, h_1}, \psi \in \tilde{\mathcal{V}}_{g_2, h_2}$, one has

$$(8.8) \quad [\varphi] \otimes [\psi] = [R_{h_1} \psi] \otimes [\varphi].$$

In particular, $[\varphi] \otimes [\psi] = [\psi] \otimes [\varphi]$ if one of $[\varphi], [\psi]$ is in $\tilde{\mathcal{V}}_{*, 1}$.

(3) For each $g \in G$, R_g is an algebra automorphism with respect to \otimes :
 $R_g(x \otimes y) = R_g(x) \otimes R_g(y)$, $R_g(\tilde{\mathbf{1}}) = \tilde{\mathbf{1}}$.

Proof. The proof is straightforward and is left to the reader. \square

Convolution product. Vector space $\tilde{\mathcal{V}}$ also has another associative product, which we will denote by $*$ (the convolution product). It is defined as follows:

8.5. Definition. Let $\varphi: V_i \rightarrow {}^{g_1}V_i, \psi: V_j \rightarrow {}^{g_2}V_j$. Then $[\varphi] * [\psi] \in \tilde{\mathcal{V}}$ is defined by

- If ${}^{g_2}V_j, V_i$ are not isomorphic, then $[\varphi] * [\psi] = 0$
- If there exists an isomorphism $\lambda: {}^{g_2}V_j \xrightarrow{\sim} V_i$, then

$$[\varphi] * [\psi] = d_i^{-1} [V_j \xrightarrow{\psi} {}^{g_2}V_j \xrightarrow{\lambda} V_i \xrightarrow{\varphi} {}^{g_1}V_i \xrightarrow{R_{g_1}(\lambda^{-1})} {}^{g_1 g_2}V_j]$$

where, as before, $d_i = \dim V_i = \dim {}^{g_2}V_j = d_j$.

It is obvious that $[\varphi] * [\psi]$ is independent of the choice of isomorphism $\lambda: {}^{g_2}V_j \xrightarrow{\sim} V_i$ and that, for $[\varphi] \in \tilde{\mathcal{V}}_{g_1, h_1}, [\psi] \in \tilde{\mathcal{V}}_{g_2, h_2}$, we have

$$\begin{aligned} [\varphi] * [\psi] &\in \tilde{\mathcal{V}}_{g_1 g_2, h_1} & h_1 &= h_2 \\ [\varphi] * [\psi] &= 0 & h_1 &\neq h_2. \end{aligned}$$

This definition is chosen so that if V is a simple object, $\psi: V \rightarrow {}^g V, \varphi: {}^g V \rightarrow {}^{hg} V$, then

$$[\varphi] * [\psi] = \frac{1}{\dim V} [\varphi \psi].$$

8.6. Lemma. (1) The product $*$ defined by Definition 8.5 defines on $\tilde{\mathcal{V}}$ a structure of an associative algebra with unit

$$(8.9) \quad \mathbf{d} = \sum_{i \in I} d_i \chi_i.$$

(2) $[\varphi] * [\psi] = [\psi] * [\varphi]$ if one of $[\varphi], [\psi]$ is in $\tilde{\mathcal{V}}_{1, *}$.

(3) For each $g \in G$, R_g is an algebra automorphism with respect to $*$: $R_g(x * y) = R_g(x) * R_g(y)$, $R_g(\mathbf{d}) = \mathbf{d}$.

The proof of this lemma is trivial and left to the reader.

It is possible to give an explicit description of the structure of $\tilde{\mathcal{V}}$ as an algebra with respect to $*$. Recall that for $i \in I$, we denoted $G_i = \text{Stab}(i)$ and that action of G on \mathcal{C} defines for every i a cohomology class $[\alpha] \in H^2(G_i, \mathbb{C}^\times)$ (see discussion preceding Theorem 3.5).

8.7. Theorem. One has a canonical isomorphism of associative algebras

$$\tilde{\mathcal{V}} = \bigoplus_{i \in I} \mathbb{C}^{\alpha^{-1}}[G_i]$$

where $\tilde{\mathcal{V}}$ is considered with respect to $*$ product, and $\mathbb{C}^{\alpha^{-1}}[G_i]$ is the twisted group algebra (3.4).

Proof. For every $i \in I, g \in G_i$, choose an isomorphism $\lambda_{i,g}: {}^g V_i \xrightarrow{\sim} V_i$. Then $\lambda_g R_g(\lambda_h) = \alpha_{gh} \lambda_{gh}$, where $\alpha \in C^2(G_i, \mathbb{C}^\times)$ is the two-cocycle from Theorem 3.5.

Rewrite this in the following form:

$$(R_g(\lambda_h^{-1}) \lambda_g^{-1} \lambda_h)(\lambda_h^{-1}) = \alpha_{gh}^{-1} \lambda_{gh}^{-1}.$$

Denoting $x_g = d_i[\lambda_g^{-1}] \in \tilde{\mathcal{V}}_{g,*}$ and using (8.3), we get $x_g * x_h = \alpha_{gh}^{-1} x_{gh}$. \square

8.8. Example. Let $\mathcal{C} = GVec$ be the category of G -graded vector spaces as in Example 5.2. Then, for any pair $g, h \in G$ such that $gh = hg$, we have a canonical isomorphism $\chi_{g,h}: X_h \rightarrow {}^g X_h$. Elements $\chi_{g,h}, gh = hg$ form a basis in $\tilde{\mathcal{V}}$. In this basis, the multiplication is given by

$$\begin{aligned} \chi_{g_1,h} * \chi_{g_2,h} &= \chi_{g_1 g_2, h} \\ \chi_{g,h_1} \otimes \chi_{g,h_2} &= \chi_{g,h_1 h_2} \end{aligned}$$

(all other products are zero).

Bilinear form. We define the bilinear form on $\tilde{\mathcal{V}}$ as follows. First, recall that for $\varphi: V \rightarrow {}^g V$, we have an adjoint morphism $\varphi^*: {}^g V^* \rightarrow V^*$. This defines on $\tilde{\mathcal{V}}$ a linear map $*$: $\tilde{\mathcal{V}}_{g,h} \rightarrow \tilde{\mathcal{V}}_{g^{-1},h^{-1}}$ such that $(\varphi \otimes \psi)^* = \psi^* \otimes \varphi^*$ and $(\varphi * \psi)^* = \psi^* * \varphi^*$.

Next, define the constant term map $[\]_0: \tilde{\mathcal{V}} \rightarrow \mathbb{C}$ as follows

$$\begin{aligned} [\varphi]_0 &= 0, \quad \varphi: V_i \rightarrow {}^g V_i, \quad i \neq 0 \\ [\chi_0^g]_0 &= 1, \end{aligned}$$

where, as before, $\chi_0^g: \mathbf{1} \rightarrow {}^g \mathbf{1}$ is the canonical isomorphism. One easily sees that it completely determines $[\]_0$ and that $[x]_0 = 0$ if $x \in \tilde{\mathcal{V}}_{g,h}, h \neq 1$.

Now define the bilinear form

$$(8.10) \quad (\varphi, \psi) = [\varphi \otimes \psi^*]_0.$$

8.9. Lemma. *The bilinear form (8.10) has the following properties:*

- (1) For $\varphi \in \tilde{\mathcal{V}}_{g_1,h_1}, \psi \in \tilde{\mathcal{V}}_{g_2,h_2}$, we have $(\varphi, \psi) = 0$ unless $g_1 = g_2^{-1}, h_1 = h_2$.
- (2) The form is symmetric: $(\varphi, \psi) = (\psi, \varphi)$, non-degenerate, and G -invariant.
- (3) $(\chi_i, \chi_j) = \delta_{ij}$
- (4) $(x \otimes y, z) = (x, z \otimes y^*)$.

The proof of this lemma is straightforward and left to the reader. Note, however, that $(\chi_i, \chi_j) = \delta_{ij}$ is not sufficient to determine $(\ , \)$.

8.10. Example. Consider the subalgebra in $\tilde{\mathcal{V}}$ generated by classes χ_0^g . Then this subalgebra, considered with $*$ product is isomorphic to the group algebra $\mathbb{C}[G]$, and considered with the \otimes product, it is isomorphic to the algebra $\mathcal{F}(G)$ of functions on G . The bilinear form $(\ , \)$ restricted to this subalgebra coincides with the standard bilinear form on $\mathbb{C}[G]$:

$$(\chi_0^g, \chi_0^h) = \delta_{g,h}.$$

Dimension homomorphism. We define the dimension homomorphism $d: \tilde{\mathcal{V}} \rightarrow \mathbb{C}$ as follows: for $\varphi \in \tilde{\mathcal{V}}_{g,h}$, we let

$$(8.11) \quad d(\varphi) = \begin{cases} \text{tr } \varphi, & g = 1 \\ 0 & g \neq 1 \end{cases}$$

As before, we have $d_i = d(\chi_i) = \dim V_i, i \in I(\mathcal{C})$. It is immediate from the definition that so defined dimension homomorphism satisfies properties similar to those in $G = \{1\}$ case with additional property of being G -invariant:

$$(8.12) \quad \begin{aligned} d(x \otimes y) &= d(x)d(y), \\ d(x^*) &= d(x), \\ d(\mathbf{1}) &= 1, \\ d(R_g \varphi) &= d(\varphi). \end{aligned}$$

8.11. Lemma. *Let $\mathbf{d} \in \tilde{\mathcal{V}}_{1,*}$ be defined by (8.9). Then*

- (1) $(\mathbf{d}, \varphi) = d(\varphi)$.
- (2) $\mathbf{d}^* = \mathbf{d}$
- (3) *For any $x \in \tilde{\mathcal{V}}_{1,*}$, $\mathbf{d} \otimes x = x \otimes \mathbf{d} = d(x)\mathbf{d}$.*
- (4) $(\mathbf{d}_h)^* = \mathbf{d}_{h^{-1}}$.
- (5) *Let $\mathbf{d}_h = \sum_{i \in I_h} d_i \chi_i \in \tilde{\mathcal{V}}_{1,h}$, so that $\mathbf{d} = \sum_h \mathbf{d}_h$. Then for any $x \in \tilde{\mathcal{V}}_{1,h}$,*

$$x \otimes \mathbf{d}_{h^{-1}} = \mathbf{d}_{h^{-1}} \otimes x = d(x)\mathbf{d}_1$$

Proof. Part (1) is immediate from the definition and $(\chi_i, \chi_j) = \delta_{ij}$ (see Lemma 8.9); part (2) is trivial. Part (3) is proved in exactly the same way as in the proof of Lemma 7.3.

Parts (4), (5) are obtained from (2), (3) respectively by writing each side as a sum of homogeneous components. \square

8.12. Lemma. *If V is a simple module, and $\varphi: V \rightarrow {}^g V, \psi: {}^g V \rightarrow V$, then*

$$(8.13) \quad ([\varphi], [\psi]) = \frac{1}{\dim V} \text{tr}(\varphi\psi).$$

Proof. Consider $\varphi \otimes \psi^*: V \otimes V^* \rightarrow {}^g V \otimes {}^g V^*$. Since V is simple, the multiplicity of $\mathbf{1}$ in $V \otimes V^*$ is one. Let $i_V: \mathbf{1} \rightarrow V \otimes V^*$ be the canonical embedding; then by definition, the product (x, y) is defined by

$$(\varphi \otimes \psi^*)i_V = ([\varphi], [\psi])i_{gV}.$$

Pairing both sides with the canonical evaluation map $e_{gV}: {}^g V \otimes {}^g V^* \rightarrow \mathbf{1}$, we get

$$d(V)([\varphi], [\psi]) = e_{gV}(\varphi \otimes \psi^*)i_V.$$

It is immediate from the definition of the adjoint morphism that the right-hand side of this identity is equal to $\text{tr}(\varphi\psi)$. \square

As before, this may be false if V is not simple.

8.13. Lemma. *For any $x, y \in \tilde{\mathcal{V}}$, one has*

$$(x, y) = d(x * y) = (\mathbf{d}, x * y).$$

Proof. Let $x \in \tilde{\mathcal{V}}_{g_1, h_1}, y \in \tilde{\mathcal{V}}_{g_2, h_2}$. If $g_1 \neq g_2^{-1}$ or $h_1 \neq h_2$, then both sides are zero. Thus, it suffices to consider the case when $x = [\varphi], y = [\psi]$ for some $\varphi: V_i \rightarrow {}^g V_i, \psi: {}^g V_j \rightarrow V_j, V_i, V_j \in \mathcal{C}_h$. In this case the result follows from Lemma 8.12. \square

Extended Verlinde algebra and orbifold category. Let us describe relation between the extended Verlinde algebra $\tilde{\mathcal{V}}(\mathcal{C})$ and the Verlinde algebra $\mathcal{V}(\mathcal{C}/G)$. Define the map $F: \mathcal{V}(\mathcal{C}/G) \rightarrow \tilde{\mathcal{V}}(\mathcal{C})$ by

$$(8.14) \quad F(X, \{\varphi_g\}) = \sum_g [\varphi_g].$$

Note that $[\varphi_g] \in \tilde{\mathcal{V}}_{g^{-1},*}$.

It is also useful to write map F in terms of the morphisms $\varphi_{g,h}: {}^hX \rightarrow {}^gX$ defined in Remark 3.2.

8.14. Lemma.

$$F(X) = \frac{1}{|G|} \sum_{g,h} [\varphi_{g,h}].$$

Proof. It suffices to prove that for any $g \in G$,

$$[\varphi_{a,b}] = [\varphi_{ag,bg}]$$

which in turn is equivalent to $[\varphi_{a,b}] = [\varphi_{1,ba^{-1}}]$. To prove this, consider the following diagram:

$$\begin{array}{ccc} {}^{ba^{-1}}X & \xrightarrow{\varphi_{1,ba^{-1}}} & X \\ \varphi_{b,ba^{-1}} \downarrow & & \downarrow \varphi_{a,1} \\ {}^bX & \xrightarrow{\varphi_{b,a}} & {}^aX \end{array}$$

By (3.3), this diagram is commutative, and $\varphi_{b,ba^{-1}} = R_b(\varphi_{1,a^{-1}})$; thus, by (8.3), $[\varphi_{a,b}] = [\varphi_{1,ba^{-1}}]$. \square

8.15. Corollary.

$$F(X) = \sum [\varphi_g^{-1}]$$

Also, define the map $G: \tilde{\mathcal{V}}(\mathcal{C}) \rightarrow \mathcal{V}(\mathcal{C}/G)$ as follows: for $\psi: V \rightarrow {}^gV$, we let

$$(8.15) \quad \begin{aligned} G[\psi] &= [f] \\ f: \text{Ind } V &\rightarrow \text{Ind } V \\ f &= \bigoplus_h R_h(\psi): {}^hV \rightarrow {}^{hg}V \end{aligned}$$

These definitions extend the maps $\mathcal{V}(\mathcal{C}/G) \rightarrow \mathcal{V}(\mathcal{C})$, $\mathcal{V}(\mathcal{C}) \rightarrow \mathcal{V}(\mathcal{C}/G)$ given by the restriction and induction functors defined by (3.5).

8.16. Example. By Example 3.6, $G[\mathbf{1}] = [\text{Ind } \mathbf{1}] = [\mathcal{F}(G) \otimes \mathbf{1}]$. More generally, for the canonical morphism $\chi_0^g: \mathbf{1} \rightarrow {}^g\mathbf{1}$ defined by (2.1), one has $G\chi_0^g = \pi_g$, where $\pi_g: [\mathcal{F}(G) \otimes \mathbf{1}] \rightarrow [\mathcal{F}(G) \otimes \mathbf{1}]$ is the right regular action of g on $\mathcal{F}(G)$. Decomposing $\mathcal{F}(G) = \bigoplus_\lambda \rho_\lambda \otimes \rho_\lambda^*$, where λ runs over the set \hat{G} of isomorphism classes of irreducible G -modules, we see that

$$G(\chi_0^g) = \bigoplus_\lambda \text{tr}_{\rho_\lambda^*}(g) \cdot [\rho_\lambda \otimes \mathbf{1}].$$

Since $\sum_g \pi_g$ acts by multiplication by $|G|$ on $\rho_0 \otimes \rho_0 \subset \mathcal{F}(G)$ and by zero on $\rho_\lambda \otimes \rho_\lambda^*$, $\lambda \neq 0$, we get

$$G(\tilde{\mathbf{1}}) = G(\sum \chi_0^g) = |G| \cdot [\mathbf{1}_{\mathcal{C}/G}]$$

Relation between maps F and G is given by the following theorem, parallel to [KO, Theorem 1.6].

8.17. Theorem.

- (1) $F(x \otimes y) = F(x) \otimes F(y)$, $G(F(x) \otimes y) = x \otimes G(y)$.
- (2) $GF(x) = |G|x$, $FG(x) = \sum_h R_h(x)$.
- (3) $F(x^*) = (F(x))^*$, $G(x^*) = (G(x))^*$, $[G(x)]_0 = [x]_0$
- (4) $(Fx, y) = (x, Gy)$

Proof. (1) For F , immediate from the definition; for G , follows by a simple explicit computation.

- (2) $FG(x) = \sum_h R_h(x)$ is immediate from the definitions; $GF(x) = |G|x$ follows from $GF(x) = G(F(x) \otimes \mathbf{1}) = x \otimes G(\mathbf{1})$ (which is a special case of part (1)) and results of Example 8.16.

- (3) For $*$, obvious; for $[\]_0$, note that if $\psi: V_i \rightarrow {}^g V_i$, $i \neq 0$, then the multiplicity $[\text{Res } V_i : \mathbf{1}_{\mathcal{C}/G}] = 0$, so $[G\psi]_0 = 0 = [\psi]_0$. For $\psi = \chi_0^g$, it follows from Example 8.16 that $[G\chi_0^g]_0 = 1 = [\chi_0^g]_0$.

- (4) Using results of previous parts,

$$\begin{aligned} (Fx, y) &= [Fx \otimes y^*]_0 = [G(Fx \otimes y^*)]_0 = [x \otimes Gy^*]_0 \\ &= (x, Gy) \end{aligned}$$

□

8.18. Corollary. *The map F is an isomorphism*

$$\mathcal{V}(\mathcal{C}/G) \xrightarrow{\sim} (\tilde{\mathcal{V}}(\mathcal{C}))^G$$

Proof. First, note that it is immediate from (3.3) and Lemma 8.14 that $\text{Im } F \subset (\tilde{\mathcal{V}}(\mathcal{C}))^G$. The fact that it is an isomorphism is immediate from part (2) of Theorem 8.17: the inverse map is given by $\frac{1}{|G|}G$. □

9. S -MATRIX

Similar to the situation in $G = \{1\}$ case (see, e.g., [BK, Chapter 3]), in this section we introduce linear operators $\tilde{s}, \tilde{t}: \tilde{\mathcal{V}} \rightarrow \tilde{\mathcal{V}}$. Later we will show that if \tilde{s} is non-degenerate, then after a simple renormalization these operators define an action of the modular group $\text{SL}_2(\mathbb{Z})$ on $\tilde{\mathcal{V}}$.

In this section, we assume that \mathcal{C} is a G -equivariant fusion category, and that the set I of isomorphism classes of simple objects in \mathcal{C} is finite. We denote by $\tilde{\mathcal{V}}$ the extended Verlinde algebra of \mathcal{C} as defined in Section 8.

Define

$$\begin{aligned} (9.1) \quad \tilde{t}: \tilde{\mathcal{V}}_{g,h} &\rightarrow \tilde{\mathcal{V}}_{gh,h} \\ [\varphi] &\mapsto [\theta\varphi] = [\varphi\theta] \end{aligned}$$

where θ is the universal twist. Equality $[\theta\varphi] = [\varphi\theta]$ follows from (8.3) and $R_g\theta = \theta$ (Lemma 2.3): if $\varphi: V \rightarrow {}^g V$, $V \in \mathcal{C}_h$, then

$$(9.2) \quad [\theta_g \varphi] = [V \xrightarrow{\varphi} {}^g V \xrightarrow{\theta} {}^{hg} V] = [{}^{h^{-1}} V \xrightarrow{R_{h^{-1}}\theta} V \xrightarrow{\varphi} {}^g V] = [\varphi\theta_{h^{-1}} V].$$

We also define

$$(9.3) \quad \begin{aligned} \tilde{s}: \tilde{\mathcal{V}}_{g,h} &\rightarrow \tilde{\mathcal{V}}_{h^{-1},g} \\ \tilde{s}[\varphi] &= \sum_{i \in I_g} (\tilde{s}[\varphi])_i \end{aligned}$$

where, for $\varphi: V \rightarrow {}^g V, V \in \mathcal{C}_h$, we define $(\tilde{s}[\varphi])_i: V_i \rightarrow {}^{h^{-1}}V_i, V_i \in \mathcal{C}_g$ by Figure 5.

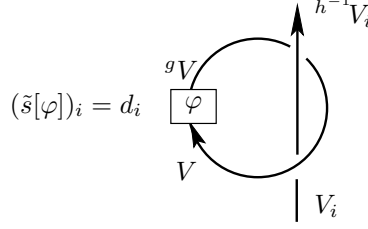
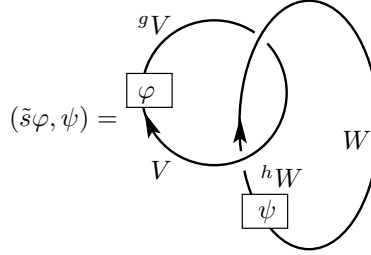


FIGURE 5. S -matrix

Equivalently, the map \tilde{s} can be described by the following lemma.

9.1. Lemma. *Let $\varphi: V \rightarrow {}^g V, V \in \mathcal{C}_h$ and $\psi: W \rightarrow {}^h W, W \in \mathcal{C}_g$, so $[\varphi] \in \tilde{\mathcal{V}}_{g,h}, [\psi] \in \tilde{\mathcal{V}}_{h,g}$. Then*



Proof. Suffices to prove in the case when W is simple, in which case it follows from Lemma 8.9(4). \square

9.2. Lemma. (1) *The restriction of \tilde{s}, \tilde{t} to $\tilde{\mathcal{V}}_{1,1} = \mathcal{V}(\mathcal{C}_1)$ coincides with the \tilde{s}, \tilde{t} matrices defined in [BK].*

(2) *s, t are symmetric: $(\tilde{s}\varphi, \psi) = (\varphi, \tilde{s}\psi), (\tilde{t}\varphi, \psi) = (\varphi, \tilde{t}\psi)$.*

(3) *$\tilde{s}\mathbf{1} = \mathbf{d}_1, \tilde{s}\tilde{\mathbf{1}} = \mathbf{d}$.*

(4) *$\tilde{s}(x \otimes y) = \tilde{s}(y) * \tilde{s}(x)$*

Proof. (1) Immediate from the definition.

(2) Symmetry of \tilde{s} is immediate from Lemma 9.1, after some simple manipulation of figures using the Reidemeister moves of Section 6. To prove symmetry of \tilde{t} , notice that by Lemma 8.12, one has

$$(\tilde{t}x, y) = d(\Theta * x * y)$$

where

$$\Theta = \sum d_i [\theta_{V_i}]$$

Thus, it suffices to prove that Θ is central with respect to $*$, which is immediate from (9.2).

(3) Immediate from the definition.

(4) Let $\varphi: V \rightarrow {}^gV, \psi: W \rightarrow {}^gW$. Using the graphical calculus of Section 6, we rewrite $\tilde{s}([\varphi]) * \tilde{s}([\psi])$ as follows:

$$\begin{aligned}
 \tilde{s}([\varphi]) * \tilde{s}([\psi]) &= \\
 &= \sum_i d_i \text{ (diagram with two circles: top circle labeled } {}^gV \text{ with box } \varphi, \text{ bottom circle labeled } {}^gW \text{ with box } \psi, \text{ both connected to } V_i \text{)} \\
 &= \sum_i d_i \text{ (diagram with two concentric circles: outer circle labeled } W \text{ with box } \psi, \text{ inner circle labeled } V \text{ with box } \varphi, \text{ both connected to } V_i \text{)} \\
 &= \sum_i d_i \text{ (diagram with one circle labeled } W \otimes V \text{ with box } \psi \otimes \varphi, \text{ connected to } V_i \text{)} \\
 &= \tilde{s}([\psi] \otimes [\varphi]).
 \end{aligned}$$

□

9.3. Theorem. *The operators \tilde{s}, \tilde{t} satisfy the following relations*

$$\begin{aligned}
 (\tilde{s}\tilde{t})^3 &= p^+ \tilde{s}^2 \\
 (\tilde{s}\tilde{t}^{-1})^3 &= p^- \tilde{s}^2 c \\
 c\tilde{t} &= \tilde{t}c, \quad c\tilde{s} = \tilde{s}c
 \end{aligned}
 \tag{9.4}$$

where $c: \tilde{\mathcal{V}} \rightarrow \tilde{\mathcal{V}}$ is defined by $c[\varphi] = [\varphi^*]$ and

$$p^\pm = \sum_{i \in I_1} \theta_i^{\pm 1} d_i^2.
 \tag{9.5}$$

Note that this sum is over I_1 , i.e. simple objects in \mathcal{C}_1 only.

To prove this, theorem, we need first to prove several preparatory results.

9.4. Proposition. *For any $h \in G, i \in I_h$,*

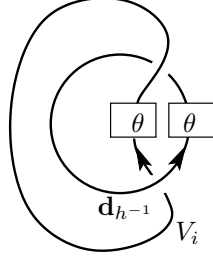
$$\text{(diagram: circle with box } \theta^{\pm 1} \text{, arrows labeled } {}^{h\pm 1}\mathbf{d}_h \text{ and } \mathbf{d}_h \text{, connected to } V_i \text{)} = p^\pm \text{id}_{V_i}$$

where \mathbf{d}_h is as in Lemma 8.11.

Proof. We prove the identity for θ ; for θ^{-1} , the proof is similar.

Since any morphism $V_i \rightarrow V_i$ is a multiple of identity, it suffices to compute the trace of the left-hand side. Replacing \mathbf{d}_h by $\mathbf{d}_h^* = \mathbf{d}_{h^{-1}}$ (see Lemma 8.11)

and reversing the direction of the corresponding strand, we see that the trace of left-hand side is given by



Using the identities $\theta_{V \otimes W} = \theta \otimes \theta R^2$ (see Lemma 2.3) and $\mathbf{d}_{h^{-1}} \otimes V_i = d_i \mathbf{d}_1$ (see Lemma 8.11), we can rewrite this as follows

$$\text{Diagram 1} = d_i \text{Diagram 2} = d_i p^+.$$

Diagram 1: A circle with a box containing θ on the right side. Below the circle is the label $\mathbf{d}_{h^{-1}} \otimes V_i$.

Diagram 2: A circle with a box containing θ on the right side. Below the circle is the label \mathbf{d}_1 .

□

9.5. Corollary. *For any $V \in \mathcal{C}_h$, one has*

$$\text{Diagram 1} = p^\pm \theta_V^{\mp 1}.$$

Diagram 1: A circle with a box containing $\theta^{\pm 1}$ on the left side. Below the circle is the label \mathbf{d}_h . A vertical line extends from the bottom of the circle, labeled V . An arrow points upwards from the top of the circle, labeled ${}^{h^{\mp 1}}V$.

Proof. Since both sides are functorial morphisms $V \rightarrow {}^{h^{\mp 1}}V$, it suffices to prove this for V being a simple module, in which case it is immediate from Proposition 9.4. □

9.6. Corollary. *For any simple modules $V_i \in \mathcal{C}_{h_1}, V_k \in \mathcal{C}_{h_2}, h_1 h_2 = h$, one has*

$$\text{Diagram 1} = \text{Diagram 2}$$

Diagram 1: A circle with a box containing θ on the left side. Below the circle is the label \mathbf{d}_h . Two vertical lines extend from the bottom of the circle, labeled V_i and V_k . Two arrows point upwards from the top of the circle.

Diagram 2: Two boxes, each containing θ^{-1} , are positioned at the top. Two strands, labeled V_i and V_k at the bottom, cross each other and then go up into the boxes.

Now we are ready to prove Theorem 9.3.

Proof of Theorem 9.3. Let us prove the first identity, $(\tilde{s}\tilde{t})^3 = p^+ \tilde{s}^2$. We rewrite it in the form

$$\tilde{s}\tilde{t}\tilde{s} = p^+ \tilde{t}^{-1} \tilde{s}\tilde{t}^{-1}.$$

By definition, for $\varphi: V_i \rightarrow {}^g V_i$, $V_i \in \mathcal{C}_h$, we have

$$(9.6) \quad \tilde{s}\tilde{t}\tilde{s}[\varphi] = \sum_{k \in I_{h^{-1}g}} d_k \sum_{j \in I_g} d_j \quad \begin{array}{c} \text{Diagram: A circle with two boxes labeled } \varphi \text{ and } \theta. \text{ The circle has three strands labeled } V_i, {}^g V_i, \text{ and } h^{-1}gV_j. \text{ A vertical strand labeled } V_k \text{ is on the right, with } g^{-1}V_k \text{ above it.} \end{array}$$

Using Reidemeister moves for framed graphs, we rewrite (9.6) in the form

$$(9.7) \quad \tilde{s}\tilde{t}\tilde{s}[\varphi] = \sum_{k \in I_{h^{-1}g}} d_k \sum_{j \in I_g} d_j \quad \begin{array}{c} \text{Diagram: A circle with two boxes labeled } \varphi \text{ and } \theta. \text{ The circle has two strands labeled } V_i \text{ and } {}^g V_i. \text{ A vertical strand labeled } V_k \text{ is on the right, with } g^{-1}V_k \text{ above it.} \end{array}$$

Using Corollary 9.6, this can be rewritten as

$$(9.8) \quad \tilde{s}\tilde{t}\tilde{s}[\varphi] = p^+ \sum_{k \in I_{h^{-1}g}} d_k \quad \begin{array}{c} \text{Diagram: A circle with two boxes labeled } \varphi \text{ and } \theta^{-1}. \text{ The circle has two strands labeled } V_i \text{ and } {}^g V_i. \text{ A vertical strand labeled } V_k \text{ is on the right, with } g^{-1}V_k \text{ above it.} \end{array} = p^+ \quad \begin{array}{c} \text{Diagram: A circle with two boxes labeled } \theta^{-1} \text{ and } \varphi. \text{ The circle has two strands labeled } V_i \text{ and } {}^g V_i. \text{ A vertical strand labeled } V_k \text{ is on the right, with } g^{-1}V_k \text{ above it.} \end{array} = p^+ (\tilde{t}^{-1} \tilde{s} \tilde{t}^{-1})[\varphi].$$

This proves the first identity of Theorem 9.3. The second is proved similarly, using Corollary 9.5 for θ^{-1} . \square

Finally, the following theorem establishes a relation between the \tilde{s} operator for the G -equivariant category \mathcal{C} and the orbifold category \mathcal{C}/G . The following theorem is an analog of [KO, Theorem 4.1]; however, the use of extended Verlinde algebra allows us to simplify the statement of this theorem.

9.7. Theorem. *Let $F: \mathcal{V}(\mathcal{C}/G) \rightarrow \tilde{\mathcal{V}}(\mathcal{C})$, $G: \tilde{\mathcal{V}}(\mathcal{C}) \rightarrow \mathcal{V}(\mathcal{C}/G)$ be defined by (8.14), (8.15). Then*

$$\begin{aligned} F\tilde{s} &= |G| \cdot \tilde{s}F & F\tilde{t} &= \tilde{t}F \\ G\tilde{s} &= \frac{1}{|G|} \tilde{s}G & G\tilde{t} &= \tilde{t}G \end{aligned}$$

Proof. We start by proving commutation relations of F, G with \tilde{t} . To prove $F\tilde{t} = \tilde{t}F$, recall that by definition, for $x = [(X, \{\varphi_g\})] \in \mathcal{V}(\mathcal{C}/G)$, one has

$$F(\tilde{t}x) = \sum_g [\theta^{C/G} \varphi_g]$$

where $\theta^{\mathcal{C}/G}$ is the twist in \mathcal{C}/G . Using definition of twist in \mathcal{C}/G (Theorem 3.9), we rewrite it as

$$F(\tilde{t}x) = \sum_g [{}^g X \xrightarrow{\varphi_g} X \xrightarrow{\theta} {}^* X \xrightarrow{\varphi_*} X]$$

where $*$ has the following meaning: if we write this sequence as a direct sum of homogeneous sequences (i.e., taking place in $\mathcal{C}_h, h \in G$), then on $\mathcal{C}_h, * = h$.

Using functoriality of θ and definition of \mathcal{C}/G , we rewrite it as follows:

$$\begin{aligned} F(\tilde{t}x) &= \sum_g [{}^g X \xrightarrow{\theta} {}^* g X \xrightarrow{R_* \varphi_g} {}^* X \xrightarrow{\varphi_*} X] \\ &= \sum_g [{}^g X \xrightarrow{\theta} {}^* g X \xrightarrow{\varphi_* g} X] = \sum_a [{}^{*-1} a X \xrightarrow{\theta} {}^a X \xrightarrow{\varphi_a} X] \\ &= \sum_a [\varphi_a \theta] = \tilde{t}(Fx). \end{aligned}$$

To prove identity $G\tilde{t} = \tilde{t}G$, recall that by definition, for $\psi: V \rightarrow {}^g V, V \in \mathcal{C}_h$, we have

$$G(\tilde{t}[\psi]) = G([\theta\psi]) = [f]$$

where $f: \bigoplus_a {}^a V \rightarrow \bigoplus_a {}^a V$ is defined by

$$\begin{aligned} \bigoplus_a {}^a V &\xrightarrow{\oplus R_a(\theta\psi)} \bigoplus_a {}^{ahg} V \xrightarrow{\text{permutation}} \bigoplus_a {}^a V \\ &= \bigoplus_a {}^a V \xrightarrow{\oplus R_a(\psi)} \bigoplus_a {}^{ag} V \xrightarrow{\theta} \bigoplus_a {}^{ahg} V \xrightarrow{\text{permutation}} \bigoplus_a {}^a V \end{aligned}$$

On the other hand, $\tilde{t}G[\psi]$ is by definition given by the class of the following morphism

$$\begin{aligned} \bigoplus_a {}^a V &\xrightarrow{\oplus R_a(\psi)} \bigoplus_a {}^{ag} V \xrightarrow{\theta^{\mathcal{C}/G}} \bigoplus_a {}^{ag} V \xrightarrow{\text{permutation}} \bigoplus_a {}^a V \\ &= \bigoplus_a {}^a V \xrightarrow{\oplus R_a(\psi)} \bigoplus_a {}^{ag} V \xrightarrow{\theta} \bigoplus_a {}^{ahg} V \xrightarrow{\text{permutation}} \bigoplus_a {}^a V \end{aligned}$$

which proves $G\tilde{t}[\psi] = \tilde{t}G[\psi]$.

To prove identities involving \tilde{s} , it suffices to prove that for any $x \in \mathcal{V}(\mathcal{C}/G), y \in \tilde{\mathcal{V}}(\mathcal{C})$ one has

$$(9.9) \quad (\tilde{s}Fx, y)_{\mathcal{C}} = \frac{1}{|G|} (\tilde{s}x, Gy)_{\mathcal{C}/G}$$

Indeed, since F and G are adjoint (see Theorem 8.17), (9.9) implies $(\tilde{s}Fx, y) = \frac{1}{|G|} (F\tilde{s}x, y)$, and thus, since the form is non-degenerate, $\tilde{s}Fx = \frac{1}{|G|} F\tilde{s}x$. Similarly, using by symmetry of \tilde{s} (see Lemma 9.2) and adjointness of F, G , we see that left-hand side of (9.9) is equal to $(Fx, \tilde{s}y) = (x, G\tilde{s}y)$, and the right-hand side is equal to $\frac{1}{|G|} (x, \tilde{s}Gy)$, so (9.9) implies $G\tilde{s}y = \frac{1}{|G|} \tilde{s}Gy$.

So let us prove (9.9). Without loss of generality, we can assume that $x = [X, \{\varphi_g\}], (X, \{\varphi_g\}) \in \mathcal{C}/G$, and $y = [\psi] \in \tilde{\mathcal{V}}_{h,g}$, where $\psi: W \rightarrow {}^h W, W \in \mathcal{C}_g, hg = gh$. In this case, using Lemma 9.1 and formula for F given in Corollary 8.15,

we see that the left-hand side of (9.9) is given by

$$(9.10) \quad (\tilde{s}Fx, y) = \text{diagram}$$

where X_h is the component of X in \mathcal{C}_h (all other components give zero contribution).

Now let us compute the right-hand side. Recall that by definition, $G[\psi] = [f]$, where $f: \text{Ind } W \rightarrow \text{Ind } W$ is defined by (8.15). Using an analog of Lemma 9.1 for usual fusion categories (which can be obtained from Lemma 9.1 by letting $G = \{1\}$), we see that

$$(\tilde{s}x, Gy) = \text{diagram}$$

where the boxed crossings are commutativity isomorphisms in \mathcal{C}/G .

Writing $X = \bigoplus X_a$, $X_a \in \mathcal{C}_a$ and $\text{Ind } W = \bigoplus {}^bW$ and using definition of the commutativity isomorphism in \mathcal{C}/G , we can rewrite it as follows:

$$(\tilde{s}x, Gy) = \sum_{a, b: a^{-1}bh=b} \text{diagram}$$

(as before, one easily sees that components with $a \neq bhb^{-1}$ give zero contribution). Replacing in this formula a by bhb^{-1} , we get

$$\begin{aligned}
(\tilde{s}x, Gy) &= \sum_b \left(\text{Diagram with } \varphi_{bgb^{-1}}^{-1}, X_{bhb^{-1}}, {}^{bh}W, R_b\psi \right) = \sum_b R_b \left(\text{Diagram with } {}^gX_h, \varphi_h^{-1}, X_h, {}^hW, \psi \right) \\
&= |G| \left(\text{Diagram with } {}^gX_h, \varphi_h^{-1}, X_h, {}^hW, \psi \right)
\end{aligned}$$

Comparing this with (9.10), we get (9.9). \square

10. MODULAR EQUIVARIANT CATEGORIES

Throughout this section, \mathcal{C} is a G -equivariant fusion category with finitely many isomorphism classes of simple objects, and $\tilde{\mathcal{V}} = \tilde{\mathcal{V}}(\mathcal{C})$ is the extended Verlinde algebra as defined in Section 8.

10.1. Definition. A G -equivariant fusion category with finitely many isomorphism classes of simple objects is called *modular* if the operator $\tilde{s}: \tilde{\mathcal{V}} \rightarrow \tilde{\mathcal{V}}$, defined by (9.3), is invertible.

This definition generalizes the well-known definition of a modular tensor category

10.2. Remark. This definition is different from the one given in [T2]. Namely, the definition of [T2] only requires that the subcategory \mathcal{C}_1 be modular. It is easy to see (see Theorem 10.4 below and discussion following it) that modularity of \mathcal{C} implies modularity of \mathcal{C}_1 but converse is not true. Thus, our definition is stronger than that of [T2].

10.3. Theorem. *Let \mathcal{C} be a G -equivariant modular category. Then:*

- (1) *The numbers p^\pm defined by (9.5) are non-zero.*
- (2) *Let*

$$(10.1) \quad s = D^{-1}\tilde{s}, \quad t = \zeta^{-1}\tilde{t}$$

where $\tilde{s}, \tilde{t}: \tilde{\mathcal{V}} \rightarrow \tilde{\mathcal{V}}$ are defined in Section 9 and

$$(10.2) \quad D = \sqrt{p^+p^-}, \quad \zeta = (p^+p^-)^{1/6}.$$

Then so defined s, t satisfy the relations of $\text{SL}_2(\mathbb{Z})$:

$$(10.3) \quad (st)^3 = s^2, \quad s^2 = c, \quad ct = tc, \quad c^2 = 1$$

where $c: \tilde{\mathcal{V}} \rightarrow \tilde{\mathcal{V}}$ is as in Theorem 9.3.

Proof. Rewriting equalities of Theorem 9.3 in the form

$$\begin{aligned}\tilde{s}\tilde{t}\tilde{s} &= p^+\tilde{t}^{-1}\tilde{s}\tilde{t}^{-1} \\ \tilde{s}\tilde{t}^{-1}\tilde{s} &= p^-\tilde{c}\tilde{s}\tilde{t} = p^-\tilde{c}\tilde{t} \cdot \tilde{s}\tilde{t}\tilde{s} \cdot \tilde{s}^{-1}\end{aligned}$$

and substituting the first equality into the second one, we get $\tilde{s}^2 = p^+p^-c$. After this, the results immediately follow from Theorem 9.3. \square

Note that the numbers ζ, D are the same as for the modular category \mathcal{C}_1 ; in other words, the central charge and rank of a G -equivariant modular category are the same as for its neutral part.

As one might expect, modularity of \mathcal{C} is closely related with modularity of the orbifold category \mathcal{C}/G and with the modularity of the untwisted sector \mathcal{C}_1 .

10.4. Theorem. *Let \mathcal{C} be a G -equivariant modular category. Then both \mathcal{C}/G and \mathcal{C}_1 are modular categories, and the numbers D, ζ defined by (10.2) are related by*

$$\begin{aligned}\zeta(\mathcal{C}_1) &= \zeta(\mathcal{C}) = \zeta(\mathcal{C}/G) \\ D(\mathcal{C}_1) &= D(\mathcal{C}) = \frac{D(\mathcal{C}/G)}{|G|}\end{aligned}$$

Proof. Modularity of \mathcal{C}_1 is immediate, as the \tilde{s} -matrix for \mathcal{C}_1 is just the restriction of \tilde{s} -operator for \mathcal{C} to $\mathcal{V}(\mathcal{C}_1) = \tilde{\mathcal{V}}_{1,1}$.

To prove modularity of \mathcal{C}/G , note that by Corollary 8.18 and Theorem 9.7, we have an embedding $F: \mathcal{V}(\mathcal{C}/G) \hookrightarrow \tilde{\mathcal{V}}(\mathcal{C})$ and $F\tilde{s} = |G| \cdot \tilde{s}F$. Thus, if $\tilde{s}x = 0$ for some $x \in \mathcal{V}(\mathcal{C}/G)$, then $Fx \in \text{Ker } \tilde{s}$ in $\tilde{\mathcal{V}}(\mathcal{C})$, which contradicts modularity of \mathcal{C} .

To prove the relation between numbers D, ζ for $\mathcal{C}, \mathcal{C}/G$, note that applying F to both sides of the identity $(\tilde{s}\tilde{t})^3 = p^+\tilde{s}^2$ for \mathcal{C}/G and using Theorem 9.7, we get $p^+(\mathcal{C}) = |G| \cdot p^+(\mathcal{C}/G)$; in a similar way we get $p^-(\mathcal{C}) = |G| \cdot p^-(\mathcal{C}/G)$. \square

A natural question is whether converse is also true, i.e. whether modularity of \mathcal{C}_1 or \mathcal{C}/G imply modularity of \mathcal{C} . One easily sees that modularity of \mathcal{C}_1 does not imply modularity of \mathcal{C} : for example, if one takes any (usual) modular category \mathcal{C} and considers it as a G -equivariant fusion category with trivial G -grading and trivial action of G , then \mathcal{C}_1 is modular, but \mathcal{C} is not a G -equivariant modular category (which can be easily seen from Lemma 10.7 below).

On the other hand, modularity of \mathcal{C}/G does imply modularity of \mathcal{C} .

10.5. Theorem. *A G -equivariant fusion category \mathcal{C} is modular iff the orbifold category \mathcal{C}/G is modular.*

Proof. By Theorem 10.4, if \mathcal{C} is modular then so is \mathcal{C}/G . Thus, we only need to prove that if \mathcal{C}/G is modular, then \mathcal{C} is modular.

Assume that \mathcal{C}/G is modular; then $\tilde{s}_{\mathcal{C}/G}^2 = D^2c$ for some non-zero $D = D(\mathcal{C}/G)$. Applying to this F , we get

$$F\tilde{s}_{\mathcal{C}/G}^2 = D^2cF = \frac{1}{|G|^2}\tilde{s}_{\mathcal{C}}^2F.$$

Applying this to $\mathbf{1} \in \mathcal{V}(\mathcal{C}/G)$ and using $F\mathbf{1} = \tilde{\mathbf{1}}$, we get

$$(10.4) \quad \tilde{s}_{\mathcal{C}}^2\tilde{\mathbf{1}} = \mu\tilde{\mathbf{1}}, \quad \mu = |G|^2D^2 \neq 0$$

(compare with [KO, Lemma 4.6]).

Let us now show that for any $x \in \widetilde{\mathcal{V}}(\mathcal{C})$, $\tilde{s}x \neq 0$. Indeed, let us compute $(\tilde{s}^2\tilde{\mathbf{1}}, x \otimes y^*)$. On one hand, by (10.4) and definition of the bilinear form, we get

$$(\tilde{s}^2\tilde{\mathbf{1}}, x \otimes y^*) = \mu(\tilde{\mathbf{1}}, x \otimes y^*) = \mu(x, y).$$

On the other hand, using Lemma 9.2, we get

$$(\tilde{s}^2\tilde{\mathbf{1}}, x \otimes y^*) = (\tilde{s}\tilde{\mathbf{1}}, \tilde{s}(x \otimes y^*)) = (\tilde{s}\tilde{\mathbf{1}}, \tilde{s}(y^*) * \tilde{s}(x))$$

Thus, if $\tilde{s}(x) = 0$, then $(\tilde{s}^2\tilde{\mathbf{1}}, x \otimes y^*) = \mu(x, y) = 0$ for all y , which contradicts non-degeneracy of the form (x, y) (Lemma 8.9). \square

10.6. Remark. Combining Theorem 10.4, Theorem 10.5, we get that if \mathcal{C}/G modular, then \mathcal{C}_1 is also modular, which is exactly a statement of [KO, Theorem 4.5] in our situation. In fact, the proof given above is parallel to the proof in [KO]; we used the bilinear form to simplify some of the arguments in [KO].

As in the usual case, modularity implies a number of remarkable properties of the category. Here are the most immediate ones.

10.7. Lemma. *Let \mathcal{C} be a modular G -equivariant category. Then*

$$|I_g| = |(I_1)^g| = |\{i \in I_1 \mid V_i \simeq {}^g V_i\}|.$$

In particular, for every g , $|I_g| \geq 1$.

Proof. In a modular G -equivariant category, s -matrix gives an isomorphism $\widetilde{\mathcal{V}}_{g,h} \xrightarrow{\sim} \widetilde{\mathcal{V}}_{h^{-1},g}$. Thus,

$$|I_g| = \dim \widetilde{\mathcal{V}}_{1,g} = \dim \widetilde{\mathcal{V}}_{g,1} = |\{i \in I_1 \mid V_i \simeq {}^g V_i\}|.$$

\square

10.8. Lemma. *For any $x, y \in \widetilde{\mathcal{V}}$, one has*

$$\begin{aligned} s(x \otimes y) &= D s(y) * s(x) \\ s(x * y) &= \frac{1}{D} s(x) \otimes s(y) \end{aligned}$$

Proof. The first identity is immediate from Lemma 9.2 and definition of s . To prove the second identity, write $x = s(a)$, $y = s(b)$ for some a, b (which is possible because s is invertible); then, using the first identity, we get

$$\begin{aligned} s(x * y) &= s(s(a) * s(b)) = \frac{1}{D} s^2(b \otimes a) = \frac{1}{D} s^2(a) \otimes s^2(b) \\ &= \frac{1}{D} s(x) \otimes s(y) \end{aligned}$$

\square

10.9. Corollary. *Let \mathcal{C} be a modular G -equivariant category. Then the two structures of associative algebra on $\widetilde{\mathcal{V}}$, defined by \otimes and $*$, are isomorphic.*

10.10. Lemma. *If $\mathcal{C}_1 = \text{Vec}$, then \mathcal{C} is equivalent as a monoidal category to the category $G\text{Vec}^\omega$ of twisted G -graded vector spaces for some $\omega \in H^3(G, \mathbb{C}^\times)$.*

Proof. It follows from Lemma 10.7 that $|I_g| = 1$ for all g ; in other words for every g , there is a unique simple object $X_g \in \mathcal{C}_g$. Since the dual of a simple object is simple, this implies $X_g^* \simeq X_{g^{-1}}$ (non-canonically).

Consider $X_g \otimes X_g^*$. This object lies in \mathcal{C}_1 , thus it must be a multiple of $\mathbf{1}$. On the other hand, for a simple object X , multiplicity of $\mathbf{1}$ in $X \otimes X^*$ is one. Thus, $X_g \otimes X_g^* = \mathbf{1}$.

Consider tensor product $X_g \otimes X_h$. Since $X_g \otimes X_h \in \mathcal{C}_{gh}$, we have $X_g \otimes X_h \simeq N_{gh} X_{gh}$ for some multiplicities $N_{gh} \in \mathbb{Z}_+$. Tensoring both sides with X_h^* and using $X_h \otimes X_h^* \simeq \mathbf{1}$, we get $X_g \simeq N_{gh} X_{gh} \otimes X_h^*$. Since X_g is simple, this implies $N_{gh} = 1$, i.e. $X_g \otimes X_h \simeq X_{gh}$. \square

This lemma, combined with Example 5.3, gives a much simpler proof of the main result of [K1] (see [K1, Corollary 5.13]). The only difference is that here, we used the assumption that \mathcal{C} is modular, whereas [K1] used the assumption that \mathcal{C}/G is modular.

10.11. Lemma. *For $x \in \tilde{\mathcal{V}}$, let L_x be the operator of left multiplication by x : $L_x y = x \otimes y$. Then*

$$s L_x s^{-1} = D_x$$

where D_x is the renormalized operator of $*$ -multiplication by $s(x)$:

$$D_x y = y * \frac{s(x)}{D}$$

Proof. This is equivalent to

$$s(x \otimes s^{-1}(y)) = \frac{y * s(x)}{D}$$

which immediately follows from Lemma 10.8. \square

This lemma is an analog of the famous Verlinde formula for the usual modular categories. Note, however, that in G -equivariant case, both \otimes and $*$ are in general non-commutative, and operator D_x is not diagonal; thus, we can not say that “ s -matrix diagonalizes fusion rules”. However, in some special cases (namely, when G is commutative and the cohomology classes α in Theorem 8.7 are trivial), $*$ is commutative and therefore we can use Lemma 10.11 to compute the fusion coefficients in $\tilde{\mathcal{V}}$. This will be described in a forthcoming paper.

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DEPARTMENT OF MATHEMATICS, SUNY AT STONY BROOK, STONY BROOK, NY 11794, USA
E-mail address: kirillov@math.sunysb.edu
URL: <http://www.math.sunysb.edu/~kirillov/>